

# OPTIMAL CAPITAL STRUCTURE WITH SCALE EFFECTS UNDER SPECTRALLY NEGATIVE LÉVY MODELS\*

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**ABSTRACT.** The optimal capital structure model with endogenous bankruptcy was first studied by Leland [20] and Leland and Toft [21], and was later extended to the spectrally negative Lévy model by Hilberink and Rogers [13] and Kyprianou and Surya [17]. This paper incorporates the scale effects by allowing the values of bankruptcy costs and tax benefits dependent on the firm's asset value. These effects have been empirically shown, among others, in Warner [25], Ang et al. [1], and Graham and Smith [12]. By using the fluctuation identities for the spectrally negative Lévy process, we obtain a candidate bankruptcy level as well as a sufficient condition for optimality. The optimality holds in particular when, monotonically in the asset value, the value of tax benefits is increasing, the loss amount at bankruptcy is increasing, and its proportion relative to the asset value is decreasing. The solution admits a semi-explicit form, and this allows for instant computation of the optimal bankruptcy levels, equity/debt/firm values and optimal leverage ratios. A series of numerical studies are given to analyze the impacts of scale effects on the default strategy and the optimal capital structure.

**Keywords:** Credit risk, optimal capital structure, spectrally negative Lévy processes, scale functions, optimal stopping

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## 1. INTRODUCTION

We revisit the Leland-Toft optimal capital structure model [21] with endogenous bankruptcy. A firm is partly financed by debt of equal seniority that is continuously retired and reissued so that its maturity profile is kept constant through time. It distributes a continuous stream of coupon payments to bondholders, on which the firm receives tax benefits. The bankruptcy triggering level is determined endogenously by the shareholders so as to maximize the firm's equity value subject to the limited liability constraint. This gives a framework for obtaining the optimal capital structure that solves the tradeoff between minimizing bankruptcy costs and maximizing tax benefits.

This model was first studied by Leland [20] and Leland and Toft [21] where they assumed geometric Brownian motion for the firm's asset value, and was later extended to a Lévy model by Hilberink and Rogers [13], Le Courtois and Quittard-Pinon [19], Kyprianou and Surya [17], and Chen and Kou [8]. By introducing jumps, it allows the value of bankruptcy costs to be stochastic, and more importantly resolves the contradictory conclusion under the continuous diffusion model that the credit spreads go to zero as the maturity decreases to zero. The problem reduces to a non-standard optimal stopping problem, and the optimal bankruptcy level can be obtained via the continuous/smooth fit principle. It was solved, in particular, for the double exponential jump diffusion process [8], for stable processes [19], and for a general spectrally negative Lévy process [13, 17].

In this paper, we add more dynamics to the Lévy model by incorporating the “scale effect” or the inhomogeneity with respect to the firm size. Despite the fascinating contributions of the aforementioned papers, there are several assumptions on the bankruptcy costs and tax benefits, which are rather artificially imposed to derive explicit/analytical solutions. This paper is aimed to relax these assumptions. Here we review the assumptions required in Leland and Toft [21] and its extensions and also give empirical evidence that motivates us to relax these assumptions.

**1.1. Bankruptcy costs.** Regarding the bankruptcy costs, it is commonly assumed that their value is proportional to the firm's asset value at the time of bankruptcy. However, this is empirically rejected, and it is widely accepted that the amount of bankruptcy costs as a ratio of the asset value is dependent on the size of the firm. Following Warner [25] and Ang et al. [1], we call this the *scale effect*.

One of the earliest empirical studies was conducted by Warner [25] where he observed this effect based on the direct bankruptcy costs of the 11 railroad companies that liquidated between 1933 and 1955. In Figure 1, we apply linear regressions to Figures 1 and 2 of [25], which plot, respectively, the bankruptcy costs and their percentages relative to the market values of the firms. Although this is not a careful analysis of these data, it is still reasonable to conclude that, while the cost tends to increase, its proportion relative to the firm value tends to decrease. Ang et al. [1] also confirmed this, observing via regression methods the strict concavity of the bankruptcy cost function based on the bankruptcy data between 1963 and 1978 in the Western District of Oklahoma. The scale effect is also consistent with the

more recent and comprehensive empirical studies conducted by Bris et al. [6] under corporate bankruptcy data in Arizona and New York between 1995 and 2001.

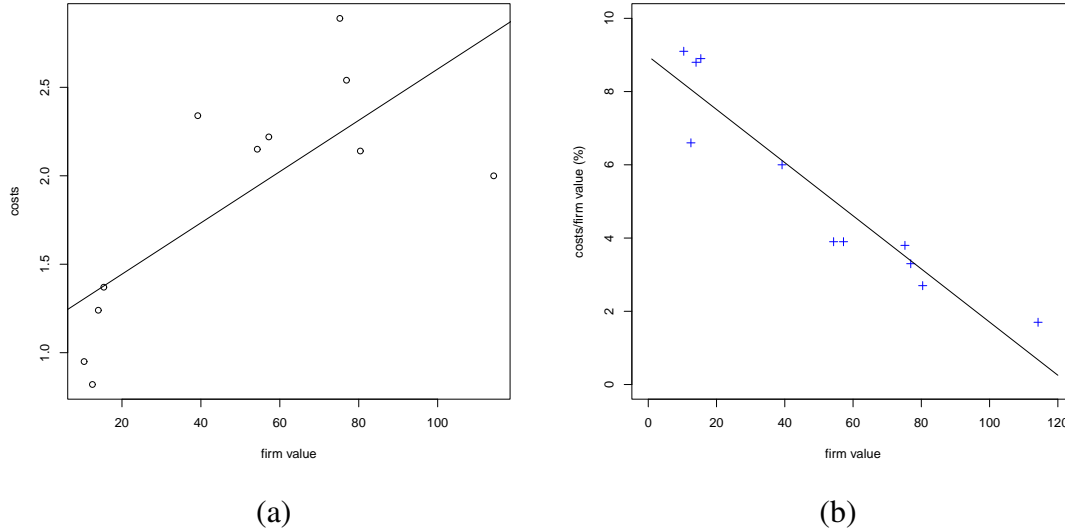


FIGURE 1. (a) Bankruptcy costs and (b) Percentage of bankruptcy costs as functions of the firm value.

These empirical results only consider the direct costs, typically consisting of legal and administrative costs. On the other hand, indirect costs are in general much harder to compute, and the estimation tends to vary heavily on the nature of the data and also on the approximation methods. While the exact amount of indirect bankruptcy costs is hard to estimate, the existence of the scale effect is easy to be validated. For example, as discussed in Franks and Torous [11] and Thorburn [24], the time spent in bankruptcy can be used as a proxy for indirect bankruptcy costs. Bris et al. [6] analyzed the mean time in bankruptcy with respect to the firm size, both for Chapter 7 liquidations and Chapter 11 reorganizations. They observed that larger bankruptcies take longer to resolve, but the relative increase is small, implying the scale effect in indirect bankruptcy costs.

Due to the difficulty of estimating the exact bankruptcy costs, it is not possible to draw a general conclusion. However, it is safe to state that the proportionality assumption of the bankruptcy costs relative to the asset value is an oversimplification, and one needs to incorporate the scale effect for realistic models. Furthermore, the characteristics of the bankruptcy costs vary heavily across industries, regions and time periods, and this further motivates us to pursue more flexible formulations.

**1.2. Tax benefits.** As for the tax benefits, empirical evidence also suggests an effect similar to the scale effect as in the case of bankruptcy costs. In other words, the marginal effect of taxable income reduction per dollar is dependent on the amount of the total taxable income that changes dynamically over time.

In the original Leland model [20] and the majority of its extensions, this dependency is ignored and a linear tax schedule, or equivalently a constant tax rate, is assumed. Because the face value and the coupon rate are fixed constant, the rate of tax benefits also stays constant over time until bankruptcy. As is clear from the fact that the tax exemption is not necessarily effective for example when the coupon payments exceed the profits, this simplification overestimates the tax benefits and result in optimal leverage that tends to be excessively higher than what is observed in reality (see, e.g., Berens and Cuny [3]).

In order to avoid this, tax cutoff is often applied. Namely, the tax rate is assumed to be a step function that is a full corporate tax rate when it is above a pre-determined level and it is zero otherwise. For spatially-homogeneous processes such as Brownian motion and Lévy processes, this can be handled relatively easily and improves the solution if the cutoff level is chosen appropriately. According to Hilberink and Rogers [13], “without such a tax cutoff, the numerical values of the coupons become ridiculously high, with the firm in effect promising huge returns financed by tax rebates; the tax cutoff prevents this”. On the other hand, they also admit that this is “an idealisation” and even a rough estimate of these cutoff levels is hard to compute.

While the tax structure is an integral factor in determining the capital structure, the estimation of the tax function, or the tax amount with respect to the taxable income, is already very difficult. Roughly speaking, income is taxable but loss is not. This implies a piecewise-linear convex tax function with zero values on the negative part and a straight line on the positive part. However, there are many factors that deform its shape. For example, for a firm facing tax progressivity, its effective marginal tax rate (or the derivative of its tax function) is monotonically increasing as opposed to being a constant. Graham and Smith [12], among others, examined the effects of statutory progressivity, net operation loss carrybacks/forwards, investment tax credits, the alternative minimum tax and uncertainty in taxable income. After these are taken into consideration, the kink at zero of the (piecewise-linear) tax function is smoothed out and results in another convex function that is strictly convex for small taxable income while it gets closer to linear as the taxable income increases (see Figures 1 and 2 of [12]). This implies that the marginal effect of taxable income reduction (or the slope of the tax function) is an increasing function that converges to the full tax rate at infinity. The effect of this convexity in the determination of the capital structure has been examined by, for example, Sarkar [22]. It is shown in their framework that the effects of tax convexity brings the optimal default boundary higher and reduces the optimal leverage.

Although there are a number of papers rectifying the overestimation of the tax benefits, most of them employ the step-function tax rate (or piecewise linear tax function) similarly to the cutoff approach described above. However, this does not fully reflect the strict convexity for small taxable income. The determination of tax benefits is heavily affected especially for firms with small and/or volatile taxable income. Furthermore, the tax-code varies across countries and across industries. For these reasons, we need a more flexible formulation of tax benefits that captures the dynamics and the tax function convexity.

This paper resolves these inflexibilities of the existing models by expressing the values of bankruptcy costs and tax benefits as functions of the firm's asset value. This generalization encompasses the existing models and adds flexibility in describing a more realistic capital structure.

We focus on the spectrally negative Lévy model considered by [13, 17] where the firm's asset value is driven by a general Lévy process with only negative jumps. Following the procedure by [17], we take advantage of the fluctuation identities expressed via the scale function. In particular, we generalize [17] using the formulas given in Egami and Yamazaki [10]. We obtain a sufficient condition for optimality of a bankruptcy strategy and show its optimality for example when, monotonically in the asset value, the value of tax benefits is increasing, the loss amount at bankruptcy is increasing, and its proportion relative to the asset value is decreasing. This condition is consistent with the empirical studies reviewed above on the bankruptcy costs and tax benefits.

The solutions to our generalized model admit semi-explicit expressions written in terms of the scale function, which has analytical forms in certain cases [10, 14, 16, 17] and can be approximated generally using, e.g., [9, 23]. In order to illustrate the implementation side, we give an example based on a mixture of Brownian motion and a compound Poisson process with i.i.d. exponential jumps. We compute the optimal bankruptcy levels and the corresponding equity/debt/firm values as well as the optimal leverage ratios as solutions to the *two-stage problem* considered in [8, 20, 21]. We conduct a sequence of numerical experiments to analyze the impacts of the scale effects on the bankruptcy strategy and the capital structure.

The rest of the paper is organized as follows. In Section 2, we review the existing Lévy model and introduce our generalized model. In Section 3, we focus on the spectrally negative Lévy model and obtain a sufficient condition for optimality. Section 4 shows examples that satisfy the sufficient condition. We give numerical results in Section 5 and concluding remarks in Section 6. All proofs are deferred to the Appendix.

## 2. PROBLEM FORMULATION

In this section, we first review the existing model and then generalize it. In particular, we adopt the formulation and notations by [13, 17] to a maximum extent. The formulation addressed here holds for any Lévy model; in the next section, we focus on the spectrally negative Lévy process and derive optimal solutions.

**2.1. The optimal capital structure model by [13, 17].** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a complete probability space hosting a Lévy process  $X = \{X_t; t \geq 0\}$ . The value of the *firm's asset* is assumed to evolve according to an *exponential Lévy process*  $V_t := e^{X_t}$ ,  $t \geq 0$ . Let  $r > 0$  be the positive risk-free interest rate and  $0 \leq \delta < r$  the total payout rate to the firm's investors. We assume that the market is arbitrage-free and that  $\mathbb{P}$  is a risk neutral probability. This requires  $\{e^{-(r-\delta)t}V_t; t \geq 0\}$  to be a  $\mathbb{P}$ -martingale. We denote by  $\mathbb{P}_x$  the probability law and  $\mathbb{E}_x$  the expectation under which  $X_0 = x$  (or equivalently  $V_0 = e^x$ ).

The firm is partly financed by debt with a constant debt profile; it issues new debt at a constant rate  $p$  with maturity profile  $\varphi(s) := me^{-ms}$  for some given constants  $p, m > 0$ . Namely, in the time interval  $(t, t + dt)$ , it issues debt with face value  $p\varphi(s)dt ds$  that matures in the time interval  $(t + s, t + s + ds)$ . By this assumption, at time 0, the face value of debt that matures in  $(s, s + ds)$  becomes

$$(2.1) \quad \left[ \int_{-\infty}^0 p\varphi(s - u) du \right] ds = pe^{-ms} ds,$$

and the face value of all debt is a constant value,

$$P := \int_0^\infty pe^{-ms} ds = \frac{p}{m}.$$

For more details, see [13, 17].

Suppose the bankruptcy is triggered at the first time  $X$  goes below a given level  $B \in \mathbb{R}$ , or

$$(2.2) \quad \tau_B^- := \inf \{t \geq 0 : X_t \leq B\}, \quad B \in \mathbb{R}.$$

The debt pays a constant coupon flow at a fixed rate  $\hat{\rho} > 0$  and a constant fraction  $0 \leq \hat{\eta} \leq 1$  of the asset value is lost at the bankruptcy time  $\tau_B^-$ ; the value of the debt with a unit face value and maturity  $t > 0$  becomes

$$(2.3) \quad d(x; B, t) := \mathbb{E}_x \left[ \int_0^{t \wedge \tau_B^-} e^{-rs} \hat{\rho} ds \right] + \mathbb{E}_x \left[ e^{-rt} 1_{\{t < \tau_B^-\}} \right] + \frac{1}{P} \mathbb{E}_x \left[ e^{-r\tau_B^- + X_{\tau_B^-}} (1 - \hat{\eta}) 1_{\{\tau_B^- < t\}} \right].$$

Here, the first term is the total value of the coupon payments accumulated until maturity or bankruptcy whichever comes first. The second term is the value of the principle payment. The last term corresponds to the  $1/P$  fraction of the asset value that is distributed, in the event of bankruptcy, to the bondholder of a unit face value. The *total value of debt* becomes, by (2.1) and Fubini's theorem,

$$\begin{aligned} \mathcal{D}(x; B) &:= \int_0^\infty pe^{-mt} d(x; B, t) dt \\ &= \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-(r+m)t} (P\hat{\rho} + p) dt \right] + \mathbb{E}_x \left[ e^{-(r+m)\tau_B^- + X_{\tau_B^-}} (1 - \hat{\eta}) 1_{\{\tau_B^- < \infty\}} \right]. \end{aligned}$$

Regarding the (*market*) *value of the firm*, it is assumed that there is a corporate tax rate  $\hat{\gamma} > 0$  and its (full) rebate on coupon payments is gained if and only if  $V_t \geq v_T$  (or  $X_t \geq \log v_T$ ) for some given cutoff level  $v_T > 0$ . Based on the Modigliani-Miller theorem (see e.g. [5]), the firm value becomes

$$\mathcal{V}(x; B) := e^x + \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-rt} 1_{\{X_t \geq \log v_T\}} P\hat{\gamma}\hat{\rho} dt \right] - \hat{\eta} \mathbb{E}_x \left[ e^{-r\tau_B^- + X_{\tau_B^-}} 1_{\{\tau_B^- < \infty\}} \right],$$

where each term corresponds to the current (unlevered) asset value, the total value of tax benefits and the value of loss at bankruptcy, respectively.

The problem is to pursue an *optimal bankruptcy level*  $B \in \mathbb{R}$  that maximizes the *equity value*,

$$(2.4) \quad \mathcal{E}(x; B) := \mathcal{V}(x; B) - \mathcal{D}(x; B), \quad x > B,$$

subject to the *limited liability constraint*,

$$(2.5) \quad \mathcal{E}(x; B) \geq 0, \quad x \geq B,$$

if such a level exists. This Lévy model was first solved by Hilberink and Rogers [13] for a special class of Lévy processes taking the form of an independent sum of a linear Brownian motion and a compound Poisson process with negative jumps (cf. (3.21) on page 245 of [13]). Kyprianou and Surya [17] later showed for a general spectrally negative process that the optimal bankruptcy level exists and is explicitly determined by applying continuous and smooth fit when  $X$  is of bounded and unbounded variation, respectively. Regarding the cases with both positive and negative jumps, Chen and Kou [8] solved for double exponential jump diffusion and Le Courtois and Quittard-Pinon [19] solved for stable processes.

**2.2. Our generalization.** We now incorporate scale effects and generalize the model described above by allowing the loss fraction  $\hat{\eta}$  and tax rebate rate  $\hat{\gamma}$  dependent on  $X$ . Although it is not of primary interest of this paper, we also generalize the coupon rate  $\hat{\rho}$  because this can be conducted in parallel and may be useful for future research; see Remark 4.1.

First, we generalize the debt value (2.3) to

$$d(x; B, t) := \mathbb{E}_x \left[ \int_0^{t \wedge \tau_B^-} e^{-rs} \rho(X_s) ds \right] + \mathbb{E}_x \left[ e^{-rt} 1_{\{t < \tau_B^-\}} \right] + \frac{1}{P} \mathbb{E}_x \left[ e^{-r\tau_B^- + X_{\tau_B^-}} \left( 1 - \bar{\eta}(X_{\tau_B^-}) \right) 1_{\{\tau_B^- < t\}} \right]$$

where  $\rho(\cdot) \geq 0$  and  $\bar{\eta}(\cdot) \geq 0$  are the coupon rate and the rate of loss at bankruptcy, respectively. The total debt value becomes

$$\mathcal{D}(x; B) := \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-(r+m)t} (P\rho(X_t) + p) dt \right] + \mathbb{E}_x \left[ e^{-(r+m)\tau_B^- + X_{\tau_B^-}} \left( 1 - \bar{\eta}(X_{\tau_B^-}) \right) 1_{\{\tau_B^- < \infty\}} \right].$$

By setting

$$f_1(y) := P\rho(y) + p \quad \text{and} \quad \eta(y) := e^y \bar{\eta}(y), \quad y \in \mathbb{R},$$

we can write

$$(2.6) \quad \mathcal{D}(x; B) = \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-(r+m)t} f_1(X_t) dt \right] + \mathbb{E}_x \left[ e^{-(r+m)\tau_B^- + X_{\tau_B^-}} 1_{\{\tau_B^- < \infty\}} \right] - \mathbb{E}_x \left[ e^{-(r+m)\tau_B^-} \eta(X_{\tau_B^-}) 1_{\{\tau_B^- < \infty\}} \right].$$

Here notice that  $\eta(\cdot)$  denotes the total loss amount whereas  $\bar{\eta}(\cdot)$  is the rate of loss relative to the asset value. We allow  $\bar{\eta}$  to be larger than 1, which lets one to model, for example, the case  $\eta$  is constant; see Section 4.3 below.



We next generalize the tax rebates; the firm value with default level  $B$  is

$$(2.7) \quad \mathcal{V}(x; B) := e^x + \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-rt} f_2(X_t) dt \right] - \mathbb{E}_x \left[ e^{-r\tau_B^-} \eta(X_{\tau_B^-}) 1_{\{\tau_B^- < \infty\}} \right],$$

where  $f_2(\cdot) \geq 0$  is the rate of tax rebates. As we discuss in Remark 3.1 below, under Assumptions 3.2-3.3, each expectation in (2.6)-(2.7) is finite for all  $x > B$  and hence  $\mathcal{E}(x; B) = \mathcal{V}(x; B) - \mathcal{D}(x; B)$  is well-defined.

### 3. SOLUTIONS FOR THE SPECTRALLY NEGATIVE LÉVY MODELS

In this section, we study the problem (2.4)-(2.5) with the debt and firm values generalized to (2.6)-(2.7) focusing on the case  $X$  is a spectrally negative Lévy process, or a Lévy process with only negative jumps. We assume that  $X$  is uniquely defined by the *Laplace exponent*,

$$(3.1) \quad \kappa(s) := \log \mathbb{E}_0 [e^{sX_1}] = cs + \frac{1}{2}\sigma^2 s^2 + \int_{(0,\infty)} (e^{-sx} - 1 + sx 1_{\{0 < x < 1\}}) \Pi(dx), \quad s \in \mathbb{R},$$

where  $c \in \mathbb{R}$ ,  $\sigma \geq 0$ , and  $\Pi$  is a measure on  $(0, \infty)$  such that

$$\int_{(0,\infty)} (1 \wedge x^2) \Pi(dx) < \infty.$$

We ignore the case  $X$  is a negative subordinator (monotonically decreasing a.s.).

The process  $X$  has paths of bounded variation if and only if  $\sigma = 0$  and  $\int_{(0,\infty)} (1 \wedge x) \Pi(dx) < \infty$ . For more details about the spectrally negative Lévy process, we refer the reader to, e.g., [4, 15].

**3.1. Scale functions.** For our derivation of optimal solutions, we first rewrite (2.6)-(2.7) using the scale function. For a given spectrally negative Lévy process with Laplace exponent  $\kappa$ , there exists an increasing function

$$W^{(q)} : \mathbb{R} \mapsto \mathbb{R}_+; \quad q \geq 0,$$

such that  $W^{(q)}(x) = 0$  for all  $x < 0$  and

$$\int_0^\infty e^{-sx} W^{(q)}(x) dx = \frac{1}{\kappa(s) - q}, \quad s > \Phi(q)$$

where

$$\Phi(q) := \sup \{s > 0 : \kappa(s) = q\}, \quad q \geq 0.$$

It is known that  $\kappa$  is zero at the origin and strictly convex on  $[0, \infty)$ . Therefore  $\Phi(q)$  is strictly increasing in  $q$ .



If  $\tau_a^+$  is the first time the process goes above  $a > x > 0$  and  $\tau_0^-$  is the first time it goes below zero as a special case of (2.2), then we have

$$\begin{aligned}\mathbb{E}_x \left[ e^{-q\tau_a^+} 1_{\{\tau_a^+ < \tau_0^-, \tau_a^+ < \infty\}} \right] &= \frac{W^{(q)}(x)}{W^{(q)}(a)}, \\ \mathbb{E}_x \left[ e^{-q\tau_0^-} 1_{\{\tau_a^+ > \tau_0^-, \tau_0^- < \infty\}} \right] &= Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)},\end{aligned}$$

where

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y) dy, \quad x \in \mathbb{R}.$$

Because  $W^{(q)}(x) = 0$  for all  $x < 0$ , we have that  $Z^{(q)}(x) = 1$  on  $(-\infty, 0]$ .

In particular,  $W^{(q)}$  is continuously differentiable on  $(0, \infty)$  if  $\Pi$  does not have atoms (see [18]), and it is twice-differentiable on  $(0, \infty)$  if  $\sigma > 0$  (see [7]). For the rest of this paper, we assume the former.

**Assumption 3.1.** *We assume that  $\Pi$  does not have atoms.*

Fix  $q > 0$ . The scale function increases exponentially;

$$(3.2) \quad W^{(q)}(x) \sim \frac{e^{\Phi(q)x}}{\kappa'(\Phi(q))} \quad \text{as } x \rightarrow \infty.$$

There exists a (scaled) version of the scale function  $W_{\Phi(q)} = \{W_{\Phi(q)}(x); x \in \mathbb{R}\}$  that satisfies

$$W_{\Phi(q)}(x) = e^{-\Phi(q)x} W^{(q)}(x), \quad x \in \mathbb{R}$$

and

$$\int_0^\infty e^{-sx} W_{\Phi(q)}(x) dx = \frac{1}{\kappa(s + \Phi(q)) - q}, \quad s > 0.$$

Moreover  $W_{\Phi(q)}(x)$  is increasing, and as is clear from (3.2),

$$W_{\Phi(q)}(x) \nearrow \frac{1}{\kappa'(\Phi(q))} \quad \text{as } x \rightarrow \infty.$$

As in Lemmas 4.3-4.4 of [17], for all  $q > 0$ ,

$$(3.3) \quad \begin{aligned} W^{(q)}(0) &= \begin{cases} 0, & \text{unbounded variation} \\ \frac{1}{\mu}, & \text{bounded variation} \end{cases}, \\ W^{(q)'}(0+) &= \begin{cases} \frac{2}{\sigma^2}, & \sigma > 0 \\ \infty, & \sigma = 0 \text{ and } \Pi(0, \infty) = \infty \\ \frac{q + \Pi(0, \infty)}{\mu^2}, & \text{compound Poisson} \end{cases}, \end{aligned}$$

where  $\mu := c + \int_{(0,1)} x \Pi(dx)$  that is finite for the case  $X$  is of bounded variation.

**3.2. In terms of the scale function.** We now rewrite (2.6)-(2.7) using the scale function. Toward this end, we introduce the following shorthand notations:

$$\Lambda^{(q)}(x; B) := \mathbb{E}_x \left[ e^{-q\tau_B^-} \eta(X_{\tau_B^-}) 1_{\{\tau_B^- < \infty\}} \right] \quad \text{and} \quad \mathcal{M}_i^{(q)}(x; B) := \mathbb{E}_x \left[ \int_0^{\tau_B^-} e^{-qt} f_i(X_t) dt \right], \quad i = 1, 2,$$

for any  $q > 0$  and  $x > B$ . These functions admit semi-explicit expressions in terms of the scale function. By Lemmas 2.1-2.3 of [10], for all  $x > B$  and  $q > 0$ , we can write

$$\begin{aligned} \Lambda^{(q)}(x; B) &= \eta(B) \left[ Z^{(q)}(x - B) - \frac{q}{\Phi(q)} W^{(q)}(x - B) \right] - W^{(q)}(x - B) H^{(q)}(B) \\ &\quad + \int_0^\infty \Pi(du) \int_0^{u \wedge (x-B)} W^{(q)}(x - z - B) [\eta(B) - \eta(z + B - u)] dz, \\ \mathcal{M}_i^{(q)}(x; B) &= W^{(q)}(x - B) G_i^{(q)}(B) - \int_B^x W^{(q)}(x - y) f_i(y) dy, \quad i = 1, 2, \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} H^{(q)}(B) &:= \int_0^\infty \Pi(du) \int_0^u e^{-\Phi(q)z} [\eta(B) - \eta(B - u + z)] dz, \\ G_i^{(q)}(B) &:= \int_0^\infty e^{-\Phi(q)y} f_i(y + B) dy, \quad i = 1, 2. \end{aligned}$$

On the other hand, by Lemma 4.7 of [17], we have  $\mathbb{E}_y \left[ e^{-q\tau_0^- + X_{\tau_0^-}} 1_{\{\tau_0^- < \infty\}} \right] = e^y - \Gamma^{(q)}(y)$  for any  $y > 0$  where

$$\Gamma^{(q)}(y) := \frac{\kappa(1) - q}{1 - \Phi(q)} W^{(q)}(y) + (\kappa(1) - q) e^y \int_0^y e^{-z} W^{(q)}(z) dz, \quad q > 0 \text{ and } y > 0. \quad (3.5)$$

Hence, for any  $x > B$ ,

$$\mathbb{E}_x \left[ e^{-(r+m)\tau_B^- + X_{\tau_B^-}} 1_{\{\tau_B^- < \infty\}} \right] = e^B \mathbb{E}_{x-B} \left[ e^{-(r+m)\tau_0^- + X_{\tau_0^-}} 1_{\{\tau_0^- < \infty\}} \right] = e^x - e^B \Gamma^{(r+m)}(x - B).$$

Putting altogether, for all  $x > B$ , we simplify (2.6)-(2.7) to

$$\begin{aligned} \mathcal{D}(x; B) &= e^x - e^B \Gamma^{(r+m)}(x - B) + \mathcal{M}_1^{(r+m)}(x; B) - \Lambda^{(r+m)}(x; B), \\ \mathcal{V}(x; B) &= e^x + \mathcal{M}_2^{(r)}(x; B) - \Lambda^{(r)}(x; B), \end{aligned}$$

and we obtain the equity value

$$\mathcal{E}(x; B) = e^B \Gamma^{(r+m)}(x - B) + (\mathcal{M}_2^{(r)}(x; B) - \Lambda^{(r)}(x; B)) - (\mathcal{M}_1^{(r+m)}(x; B) - \Lambda^{(r+m)}(x; B)). \quad (3.6)$$

In view of (3.4), the following assumption guarantees the finiteness of  $\mathcal{M}_1^{(r+m)}(x; B)$  and  $\mathcal{M}_2^{(r)}(x; B)$  for all  $x > B$ .

**Assumption 3.2.** *We assume that*

$$\int_0^\infty e^{-\Phi(r+m)y} f_1(y) dy < \infty \quad \text{and} \quad \int_0^\infty e^{-\Phi(r)y} f_2(y) dy < \infty.$$

Regarding  $\eta$ , we assume the following for the rest of this section.

**Assumption 3.3.** *We assume that  $\eta$  is  $C^2(\mathbb{R})$  and is bounded on  $(-\infty, B]$  for any fixed  $B \in \mathbb{R}$ .*

Here the  $C^2$  assumption is imposed for simplicity of the arguments; this can be relaxed as discussed in Remark 3.6 below.

**Remark 3.1.** *By Assumptions 3.2-3.3, the equity value  $\mathcal{E}(x; B)$  is well-defined for any  $x > B$ .*

**Remark 3.2.** *Because  $\eta$  is continuous,  $\Lambda^{(q)}(x; B)$  and  $\mathcal{M}_i^{(q)}(x; B)$  are continuous in  $B$  on  $(-\infty, x]$  for any fixed  $x \in \mathbb{R}$ .*

**3.3. Derivative with respect to  $B$ .** To derive the candidate bankruptcy level, we use the results in Egami and Yamazaki [10] and obtain the derivative of  $\mathcal{E}(x; B)$  with respect to  $B$ . Define

$$(3.7) \quad \Theta^{(q)}(x) := W^{(q)'}(x) - \Phi(q)W^{(q)}(x) = e^{\Phi(q)x}W'_{\Phi(q)}(x), \quad x > 0 \text{ and } q > 0,$$

which is always positive. Also as in [17],  $\Theta^{(q)}(x)$  is monotonically decreasing in  $q$  for every fixed  $x > 0$ ; see [17] for an interpretation of  $\Theta^{(q)}$  as the resolvent measure of the ascending ladder height process of  $X$ .

The derivatives of  $\Lambda^{(q)}(x; B)$  and  $\mathcal{M}_i^{(q)}(x; B)$  with respect to  $B$  require technical details. However, as shown by [10], each term can be expressed as a product of  $\Theta^{(q)}(x - B)$  and some function of  $B$  (that is independent of  $x$ ). For the proof of the following lemma, see the proof of Proposition 3.1 of [10].

**Lemma 3.1.** *For every  $x > B$  and  $q > 0$ , we have*

$$\begin{aligned} \frac{\partial}{\partial B} \Lambda^{(q)}(x; B) &= \Theta^{(q)}(x - B) \left[ \frac{q}{\Phi(q)} \eta(B) + H^{(q)}(B) + \frac{\sigma^2}{2} \eta'(B) \right], \\ \frac{\partial}{\partial B} \mathcal{M}_i^{(q)}(x; B) &= -\Theta^{(q)}(x - B) G_i^{(q)}(B), \quad i = 1, 2. \end{aligned}$$

For the derivative of (3.6) with respect to  $B$ , we further obtain the following.

**Lemma 3.2.** *For every  $x > B$ ,*

$$\frac{\partial}{\partial B} (e^B \Gamma^{(r+m)}(x - B)) = -\frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} e^B \Theta^{(r+m)}(x - B).$$

By combining the two lemmas above, we obtain the derivative of  $\mathcal{E}(x; B)$  with respect to  $B$ . For all  $B \in \mathbb{R}$ , define

$$(3.8) \quad J^{(r,m)}(B) := \left( \frac{r + m}{\Phi(r + m)} - \frac{r}{\Phi(r)} \right) \eta(B) - (H^{(r)}(B) - H^{(r+m)}(B))$$

and

$$(3.9) \quad \begin{aligned} K_1^{(r,m)}(B) &:= \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} e^B - G_1^{(r+m)}(B) + G_2^{(r)}(B) - J^{(r,m)}(B), \\ K_2^{(r)}(B) &:= G_2^{(r)}(B) + \frac{r}{\Phi(r)} \eta(B) + H^{(r)}(B) + \frac{\sigma^2}{2} \eta'(B). \end{aligned}$$

**Proposition 3.1.** *For every  $x > B$ ,*

$$(3.10) \quad \frac{\partial}{\partial B} \mathcal{E}(x; B) = - \left[ \Theta^{(r+m)}(x-B) K_1^{(r,m)}(B) + \{ \Theta^{(r)}(x-B) - \Theta^{(r+m)}(x-B) \} K_2^{(r)}(B) \right].$$

**Remark 3.3.** *If  $\eta(B)$  is increasing in  $B$ , then  $K_2^{(r)}$  is uniformly positive.*

The following remark will be especially useful in our analysis; the proof is given in the Appendix.

**Remark 3.4.** *We can also write*

$$J^{(r,m)}(B) = \frac{1}{2} \sigma^2 (\Phi(r+m) - \Phi(r)) \eta(B) + \int_0^\infty \Pi(du) \int_0^u (e^{-\Phi(r)z} - e^{-\Phi(r+m)z}) \eta(B-u+z) dz.$$

Consequently, because  $\Phi(q)$  is increasing in  $q$  and  $\eta(\cdot)$  is positive by assumption,  $J^{(r+m)}(B) \geq 0$  for all  $B \in \mathbb{R}$ .

**3.4. Continuous fit.** Before discussing the optimality, we consider the continuous fit condition:

$$\mathcal{E}(B+; B) = 0.$$

By taking  $x \downarrow 0$  in (3.6),

$$(3.11) \quad \mathcal{E}(B+; B) = e^B \Gamma^{(r+m)}(0) + (\mathcal{M}_2^{(r)}(B+; B) - \Lambda^{(r)}(B+; B)) - (\mathcal{M}_1^{(r+m)}(B+; B) - \Lambda^{(r+m)}(B+; B)).$$

Here we have by (3.5)

$$\Gamma^{(r+m)}(0) = \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} W^{(r+m)}(0),$$

and by Proposition 3.2 of [10], for both  $q = r$  and  $q = r+m$ ,

$$(3.12) \quad \begin{aligned} \Lambda^{(q)}(B+; B) &= -W^{(q)}(0) \left( \frac{q}{\Phi(q)} \eta(B) + H^{(q)}(B) \right) + \eta(B), \\ \mathcal{M}_i^{(q)}(B+; B) &= W^{(q)}(0) G_i^{(q)}(B), \quad i = 1, 2. \end{aligned}$$

Substituting (3.12) in (3.11) and because  $W^{(r)}(0) = W^{(r+m)}(0)$  as in (3.3), we obtain

$$(3.13) \quad \mathcal{E}(B+; B) = W^{(r+m)}(0) K_1^{(r,m)}(B).$$

By (3.3), we conclude that for the bounded variation case the continuous fit condition is equivalent to  $K_1^{(r,m)}(B) = 0$ , while for the unbounded variation case it always holds no matter how  $B$  is chosen.

**Remark 3.5.** *One can further pursue smooth fit for the case  $X$  is of unbounded variation. However, we do not discuss it here because it is not necessary for the proof of optimality in the subsequent sections. In fact, it can be expected that smooth fit condition  $\mathcal{E}'(B+; B) = 0$  is equivalent to  $K_1^{(r,m)}(B) = 0$ , and the optimal solution is expected to satisfy smooth fit (at least when  $\sigma > 0$  by the results obtained in [10]). Our numerical results in Section 5 verifies that this is indeed so.*

**3.5. Optimality.** We assume that there exists  $B^*$  such that

$$(3.14) \quad K_1^{(r,m)}(B) \geq 0 \iff B \geq B^*,$$

$$(3.15) \quad K_2^{(r)}(B) \geq 0, \quad B \geq B^*,$$

and prove its optimality. We later discuss sufficient conditions that guarantee (3.14)-(3.15). Notice here that (3.15) always holds given  $\eta(B)$  is increasing in view of Remark 3.3.

We first show via continuous fit and (3.14)-(3.15) that any feasible bankruptcy level must be at least as large as  $B^*$ . Toward this end, we use the following lemma.

**Lemma 3.3.** *When  $X$  is of unbounded variation, we have  $\Theta^{(r)}(y) - \Theta^{(r+m)}(y) \rightarrow 0$  as  $y \downarrow 0$ .*

**Lemma 3.4.** *Suppose there exists  $B^*$  such that (3.14) holds. If  $B$  satisfies (2.5), then  $B \in [B^*, \infty)$ .*

Now, by how  $B^*$  is chosen, (3.14)-(3.15) and the positivity of both  $\Theta^{(r)}(y) - \Theta^{(r+m)}(y)$  and  $\Theta^{(r)}(y)$  for any  $y > 0$ , Proposition 3.1 implies

$$\frac{\partial}{\partial B} \mathcal{E}(x; B) < 0, \quad B^* \leq B < x.$$

Moreover,  $B^*$  satisfies the limited liability constraint (2.5). Indeed, for any arbitrary  $x > B^*$ , we have by (3.13)

$$0 \leq W^{(r+m)}(0)K_1^{(r,m)}(x) = \mathcal{E}(x+; x) < \mathcal{E}(x; B^*).$$

Here the first inequality holds because  $K_1^{(r,m)}(x) \geq 0$  for any  $x > B^*$  by (3.14) and in particular holds by equality for the unbounded variation case. This together with the lemma above shows the optimality of  $B^*$ . In summary, we have the following.

**Theorem 3.1.** *If there exists  $B^*$  such that (3.14)-(3.15) hold, then  $B^*$  is the optimal bankruptcy level.*

**Remark 3.6.** *The assumption that  $\eta$  is twice-differentiable on  $\mathbb{R}$  as in Assumption 3.3 can be relaxed. Its twice-differentiability at a fixed  $B \in \mathbb{R}$  is required for Lemma 3.1. In view of the arguments in this section (and in particular Remark 3.2), we only need it to hold Lebesgue-a.e. as long as  $\eta$  is continuous.*

## 4. SUFFICIENT CONDITIONS

In the last section, we showed that the conditions (3.14)-(3.15) guarantee the optimality of the bankruptcy level  $B^*$ . Here we obtain more concrete and economically sound conditions that satisfy (3.14)-(3.15). We first show that Assumption 4.1 below is sufficient and also encompasses the model by [13, 17] as reviewed in Section 2.1. We then give another example with constant  $\eta$  that does not satisfy Assumption 4.1 but nonetheless guarantees the optimality. It is emphasized here that the assumptions discussed in this section are sufficient and clearly not necessary; (3.14)-(3.15) are expected to hold more generally.

**4.1. A sufficient condition.** We show that the following assumption guarantees (3.14)-(3.15) and hence the optimality of  $B^*$  holds by Theorem 3.1.

**Assumption 4.1.** *We suppose (1)  $\eta$  is increasing, (2)  $\bar{\eta}$  is decreasing, (3)  $f_1$  is decreasing, (4)  $f_2$  is increasing, and (5)  $0 \leq \bar{\eta}(\cdot) \leq 1$ .*

For the purpose of this paper, we assume that the coupon rate is constant and consider only the scale effects on tax benefits and bankruptcy costs. This makes the function  $f_1$  constant and hence Assumption 4.1 only requires (1), (2), (4) and (5).

Each condition in the assumption is justified by the empirical studies as we described in Subsections 1.1 and 1.2. The monotonicity of  $\eta$  and  $\bar{\eta}$  in (1) and (2) means that, as the firm's asset value increases, the amount of total bankruptcy costs increases and its proportion relative to the asset value decreases. This is exactly the same as the results on scale effects derived by [1, 6, 25]; see Section 1 and in particular Figure 1. The monotonicity of  $f_2$  (or the tax rebate rate) in (4) is implied by the convexity of the tax function that has been empirically confirmed by Graham and Smith [12]; the marginal effect of taxable income reduction (or the slope of the tax function) is an increasing function that converges to the full tax rate at infinity. The last condition (5) requires that the bankruptcy costs should not be more than the total asset value.

We first note that (1) guarantees (3.15) by Remark 3.3. The following proposition shows that  $K_1^{(r,m)}(B)$  is monotonically increasing and hence (3.14) also holds.

**Proposition 4.1.** *Suppose Assumption 4.1 holds. Then the unique root  $B^*$  of  $K_1^{(r,m)}(B) = 0$  satisfies (3.14)-(3.15), and it is an optimal bankruptcy level.*

In order to prove Proposition 4.1, we first rewrite, in view of (3.9) and Remark 3.4,

$$K_1^{(r,m)}(B) = e^B l(B) - G_1^{(r+m)}(B) + G_2^{(r)}(B)$$

where

$$(4.1) \quad l(B) := \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} - \frac{1}{2}\sigma^2(\Phi(r+m) - \Phi(r))\bar{\eta}(B) \\ - \int_0^\infty \Pi(du) \int_0^u (e^{-\Phi(r)z} - e^{-\Phi(r+m)z})e^{z-u}\bar{\eta}(B-u+z)dz.$$

Because  $\bar{\eta}(B)$  is decreasing in  $B$  by assumption and  $\Phi(q)$  is increasing in  $q$ ,  $l(B)$  is increasing. Because  $\bar{\eta}(B)$  is monotone and bounded in  $[0, 1]$ , there exists  $0 \leq \bar{\eta}(-\infty) := \lim_{B \downarrow -\infty} \bar{\eta}(B) \leq 1$ . By the monotone convergence theorem, we obtain

$$\begin{aligned} \lim_{B \downarrow -\infty} l(B) &= \frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} - \frac{1}{2} \sigma^2 (\Phi(r + m) - \Phi(r)) \bar{\eta}(-\infty) \\ &\quad - \int_0^\infty \Pi(du) \int_0^u (e^{-\Phi(r)z} - e^{-\Phi(r+m)z}) e^{z-u} \bar{\eta}(-\infty) dz \\ &= \frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} - \bar{\eta}(-\infty) j^{(r,m)} \end{aligned}$$

where

$$j^{(r,m)} := \frac{1}{2} \sigma^2 (\Phi(r + m) - \Phi(r)) + \int_0^\infty \Pi(du) e^{-u} \left( \frac{1 - e^{-(\Phi(r)-1)u}}{\Phi(r) - 1} - \frac{1 - e^{-(\Phi(r+m)-1)u}}{\Phi(r + m) - 1} \right).$$

**Lemma 4.1.** *We have*

$$j^{(r,m)} = \frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} - \frac{\kappa(1) - r}{1 - \Phi(r)}.$$

Now by Lemma 4.1,

$$\lim_{B \downarrow -\infty} l(B) = (1 - \bar{\eta}(-\infty)) \frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} + \bar{\eta}(-\infty) \frac{\kappa(1) - r}{1 - \Phi(r)}.$$

Because  $0 \leq \bar{\eta}(-\infty) \leq 1$  and  $\kappa$  is convex on  $[0, \infty)$  and zero at the origin, we have  $\lim_{B \downarrow -\infty} l(B) \geq 0$  and hence  $l(B) \geq 0$  for any  $B \in \mathbb{R}$ . Consequently,  $e^{Bl(B)}$  is increasing in  $B$ . Finally,  $-G_1^{(r+m)}(B) + G_2^{(r)}(B)$  is increasing in  $B$  because  $f_1$  is decreasing and  $f_2$  is increasing by assumption. Therefore, there exists a unique  $B^*$  that satisfies (3.14). As discussed above, (3.15) holds by the monotonicity of  $\eta$ . Now, by Theorem 3.1, Proposition 4.1 holds.

**Remark 4.1.** *Although it is not of our primary interest to generalize the coupon rate in this paper, Theorem 3.1 and/or Proposition 4.1 may be applicable to achieve this. In the setting of Leland-Toft and also in this paper, the firm is assumed to keep the amount of debt-financing constant over time. However, this is rather unrealistic and also suboptimal from the firm's point of view. The generalization of the coupon rate as a function may alleviate this issue. We leave this as a potential future research because it is significantly difficult. It is unlikely that it admits a solution applicable to a general spectrally negative Lévy process unlike the results obtained for bankruptcy costs and tax benefits. In comparison to the bankruptcy cost and tax rate functions, the coupon rate function is very difficult to formulate. In addition, it affects both  $f_1$  and  $f_2$  and its generalization is much more restricted than those of the bankruptcy costs and tax benefits. Nevertheless, this is an important problem and certainly the results obtained in this paper may be useful for future research.*



**4.2. Reduction to the case by [13, 17].** As an example that satisfies Assumption 4.1, we revisit the simple case by [13, 17] as reviewed in Section 2.1. Namely, we set

$$f_1(x) = P\hat{\rho} + p, \quad f_2(x) = 1_{\{x \geq \log v_T\}} P\hat{\gamma}\hat{\rho}, \quad \text{and} \quad \bar{\eta}(B) = \hat{\eta},$$

and confirm that our result matches that of [13, 17].

First, Assumption 4.1 is trivially satisfied and hence the optimal threshold level  $B^*$  is uniquely given by  $K_1^{(r,m)}(B^*) = 0$ . In this case,

$$(4.2) \quad G_1^{(r+m)}(B) = \int_0^\infty e^{-\Phi(r+m)y} f_1(y+B) dy = \frac{P\hat{\rho} + p}{\Phi(r+m)} = \frac{P(\hat{\rho} + m)}{\Phi(r+m)}$$

and

$$\begin{aligned} G_2^{(r)}(B) &= \int_0^\infty e^{-\Phi(r)y} f_2(y+B) dy = P\hat{\gamma}\hat{\rho} \int_{(\log v_T - B) \vee 0}^\infty e^{-\Phi(r)y} dy = \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)} e^{-\Phi(r)((\log v_T - B) \vee 0)} \\ &= \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)} (e^{\log v_T - B} \vee 1)^{-\Phi(r)} = \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)} ((e^{-B} v_T) \vee 1)^{-\Phi(r)} = \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)} \left( \frac{e^B}{v_T} \wedge 1 \right)^{\Phi(r)}. \end{aligned}$$

We also have by Remark 3.4 and Lemma 4.1

$$\begin{aligned} J^{(r,m)}(B) &= \frac{1}{2} \sigma^2 (\Phi(r+m) - \Phi(r)) e^B \hat{\eta} + \int_0^\infty \Pi(du) \int_0^u (e^{-\Phi(r)z} - e^{-\Phi(r+m)z}) e^{B-u+z} \hat{\eta} dz \\ &= j^{(r,m)} \hat{\eta} e^B = \left( \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} - \frac{\kappa(1) - r}{1 - \Phi(r)} \right) \hat{\eta} e^B. \end{aligned}$$

Combining the above,

$$\begin{aligned} K_1^{(r,m)}(B) &= \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} e^B - \frac{P(\hat{\rho} + m)}{\Phi(r+m)} + \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)} \left( \frac{e^B}{v_T} \wedge 1 \right)^{\Phi(r)} - \hat{\eta} e^B \left( \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} - \frac{\kappa(1) - r}{1 - \Phi(r)} \right) \\ &= -\frac{P(\hat{\rho} + m)}{\Phi(r+m)} + \frac{P\hat{\gamma}\hat{\rho}}{\Phi(r)} \left( \frac{e^B}{v_T} \wedge 1 \right)^{\Phi(r)} + e^B \left( (1 - \hat{\eta}) \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} + \hat{\eta} \frac{\kappa(1) - r}{1 - \Phi(r)} \right). \end{aligned}$$

The unique value of  $B$  that satisfies  $K_1^{(r,m)}(B) = 0$  indeed matches the result of [13, 17].

**4.3. Other examples.** Although Assumption 4.1 is already a reasonable assumption, it is expected that (3.14)-(3.15) hold more generally. As an example Assumption 4.1 is violated but the optimality of  $B^*$  holds, we consider the case the value of bankruptcy costs is a constant, i.e.,  $\eta \equiv \eta_0$ ; see [20]. In this case, we have  $\bar{\eta}(y) = \eta_0 e^{-y}$ , which violates Assumption 4.1-(5). Nonetheless, the optimality trivially holds upon the monotonicity of  $G_1^{(r+m)}$  and  $G_2^{(r)}$ . Indeed,  $H^{(r)} \equiv H^{(r+m)} \equiv 0$  and hence, by (3.8),  $J^{(r,m)}(B) = \left( \frac{r+m}{\Phi(r+m)} - \frac{r}{\Phi(r)} \right) \eta_0$ , which is a constant. Now in view of the definition of  $K_1^{(r+m)}$  in (3.9), it is clearly increasing in  $B$  for example when Assumption 4.1-(3,4) hold, and hence (3.14) is valid. Moreover, (3.15) trivially holds by Remark 3.3.

## 5. NUMERICAL STUDIES

In this section, we conduct a series of numerical experiments using the results obtained in the previous sections. Due to the generality of our formulation, there are a great number of experiments of interest. However, we rather focus on analyzing the impacts of scale effects on the capital structure because this is the main theme of this paper. For other numerical experiments such as the impacts of the Lévy measure and term structure, we refer the reader to a comprehensive study conducted by Chen and Kou [8]. We first illustrate how to compute the optimal bankruptcy level, the associated equity/debt/firm values as well as the optimal face value as a solution to the two-stage problem [8, 20, 21]. We then study computationally the impacts of the scale effects on these values. We use an example where Assumption 4.1 holds and, for  $X$ , we follow [13] and use a mixture of Brownian motion and a compound Poisson process with i.i.d. exponential jumps. For the coupon rate we assume it is constant with  $f_1 = P\hat{\rho} + p$ .

For the bankruptcy costs, let

$$(5.1) \quad \bar{\eta}(x) = \eta_0 (1 \wedge e^{-a(x-b)}), \quad x \in \mathbb{R},$$

for some  $0 \leq a \leq 1$ ,  $b \in \mathbb{R}$  and  $0 \leq \eta_0 \leq 1$ . This is clearly decreasing in  $x$  and bounded in  $[0, 1]$ . Moreover,

$$(5.2) \quad \eta(x) = e^x \eta_0 (1 \wedge e^{-a(x-b)}) = \eta_0 (e^x \wedge e^{(1-a)x+ab})$$

and

$$\eta'(x) = \begin{cases} \eta_0(1-a)e^{(1-a)x+ab}, & x > b, \\ \eta_0 e^x, & x < b, \end{cases}$$

and hence it is increasing. We call  $a$  the *degree of bankruptcy cost concavity*. The scale effect vanishes when  $a = 0$  and the ratio  $\bar{\eta}$  becomes a constant. However, as  $a$  gets larger, the concavity of the function  $\eta$  increases. When  $a = 1$ ,  $\eta(x) = \eta_0 e^{x \wedge b}$  and is constant uniformly on  $[b, \infty)$ . Clearly, the bankruptcy cost is monotonically decreasing in  $a$ .

For the tax benefits, let

$$(5.3) \quad f_2(x) = P\hat{\gamma}\hat{\rho} (e^{x-c} \wedge 1)$$

for some  $c \in \mathbb{R}$ . While it is rather an oversimplification, this efficiently models the effective tax function, empirically obtained by Graham and Smith [12], which is strictly convex for small taxable income but is closer to linear for large taxable income. Indeed, (5.3) is increasing on  $(-\infty, c]$  and is constant on  $[c, \infty)$ , making its antiderivative a desired convex function. We call  $c$  the *degree of tax convexity*. The value of tax benefits decreases monotonically as  $c$  increases.

Our choice of (5.1)-(5.3) and a constant coupon rate clearly satisfy all the conditions in Assumption 4.1 and hence Proposition 4.1 holds.

Regarding  $X$ , we consider the case  $\sigma > 0$  and jumps are of exponential type with Lévy measure

$$(5.4) \quad \Pi(du) = \lambda \beta e^{-\beta u} du, \quad u > 0.$$

Its Laplace exponent (3.1) is given by

$$\kappa(s) = \mu s + \frac{1}{2} \sigma^2 s^2 + \lambda \left( \frac{\beta}{\beta + s} - 1 \right), \quad s \in \mathbb{R}.$$

The scale function of this process has an explicit expression written in terms of a sum of exponential functions; see e.g. [9]. By straightforward but tedious algebra, we can obtain each functional in the equity value (3.6) as well as the function  $K_1^{(r,m)}$  in (3.9) that determines the optimal bankruptcy level. For their explicit forms, see Appendix A.

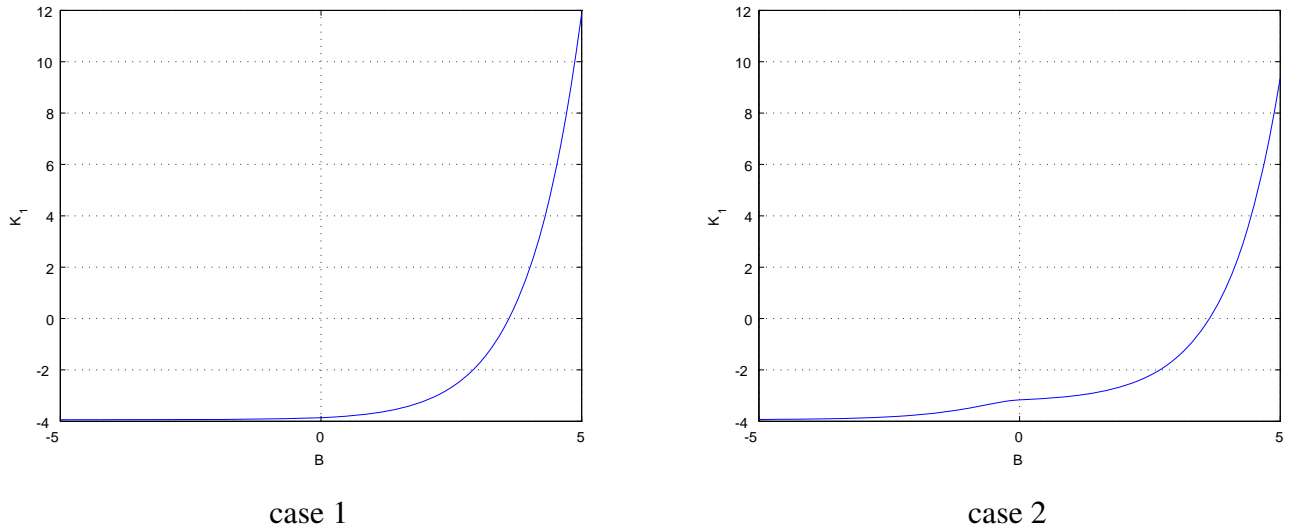


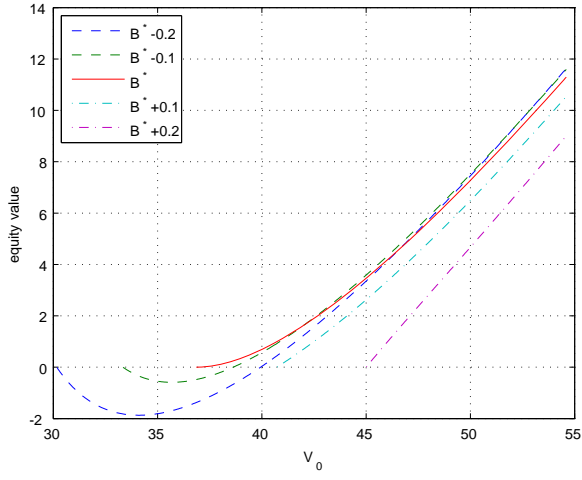
FIGURE 2. The plots of  $K_1^{(r,m)}(B)$ . The unique root of  $K_1^{(r,m)}(B) = 0$  becomes the optimal bankruptcy level  $B^*$ .

We first illustrate how to compute the optimal bankruptcy level, firm/equity/debt values and the optimal capital structure. We use  $r = 7.5\%$ ,  $\delta = 7\%$ ,  $\hat{\gamma} = 35\%$ ,  $\sigma = 0.2$ ,  $\lambda = 0.5$  and  $\beta = 9$  which were used in [13, 17, 20, 21]. We also use  $\hat{\rho} = 8.162\%$  and  $m = 0.2$ , which were used in [8]. We choose the drift term  $\mu$  so that the martingale property  $\kappa(1) = r - \delta$  is satisfied. Regarding the parameters for  $\eta$  and  $f_2$  defined above, we consider the following two cases:

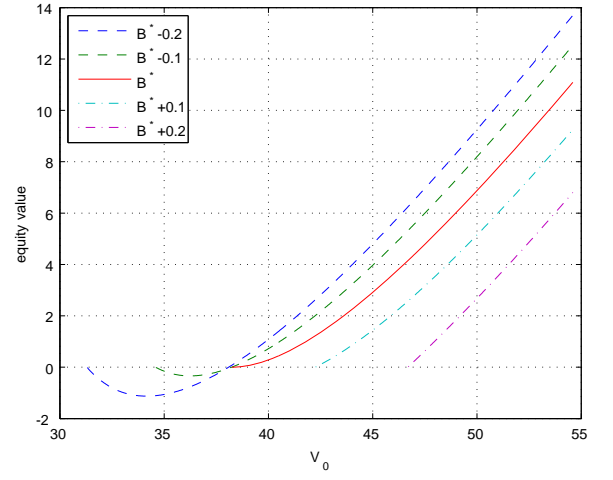
**case 1:**  $\eta_0 = 0.9$ ,  $a = 0.5$ ,  $b = 0$  and  $c = 5$ ,

**case 2:**  $\eta_0 = 0.5$ ,  $a = 0.01$ ,  $b = 5$  and  $c = 0$ .

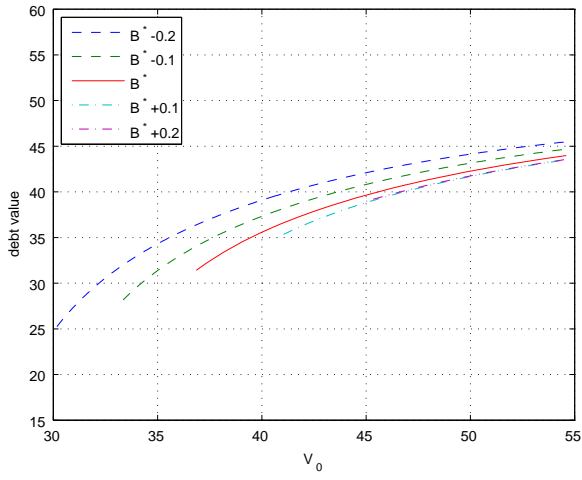
In case 1, the slopes of  $\bar{\eta}$  and  $f_2$  are magnified by how the parameters are chosen. On the other hand, in case 2, these values are constant at least when  $x \in [0, 5]$ , making the model similar to [13, 17].



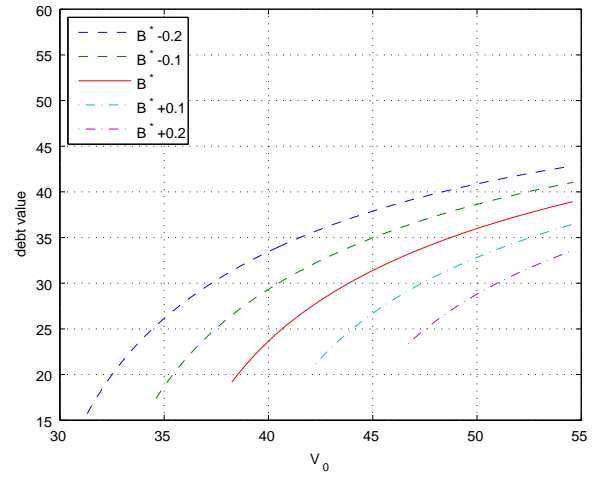
equity value (case 1)



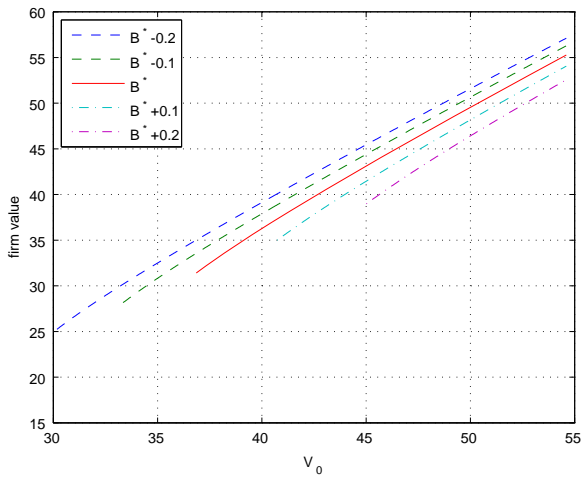
equity value (case 2)



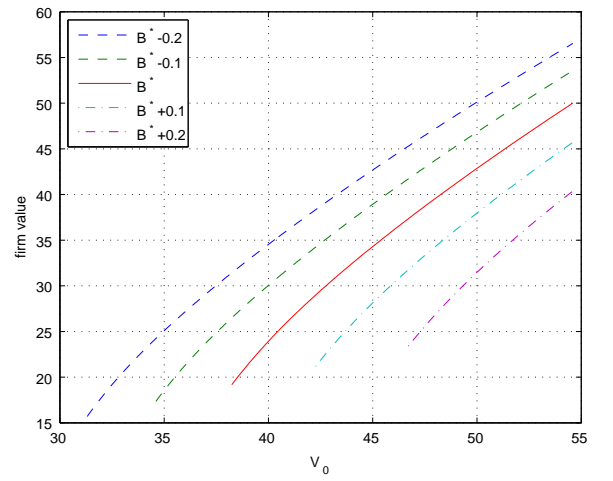
debt value (case 1)



debt value (case 2)



firm value (case 1)



firm value (case 2)

FIGURE 3. The equity/debt/firm values as a function of  $V_0$  for various values of  $B$ .

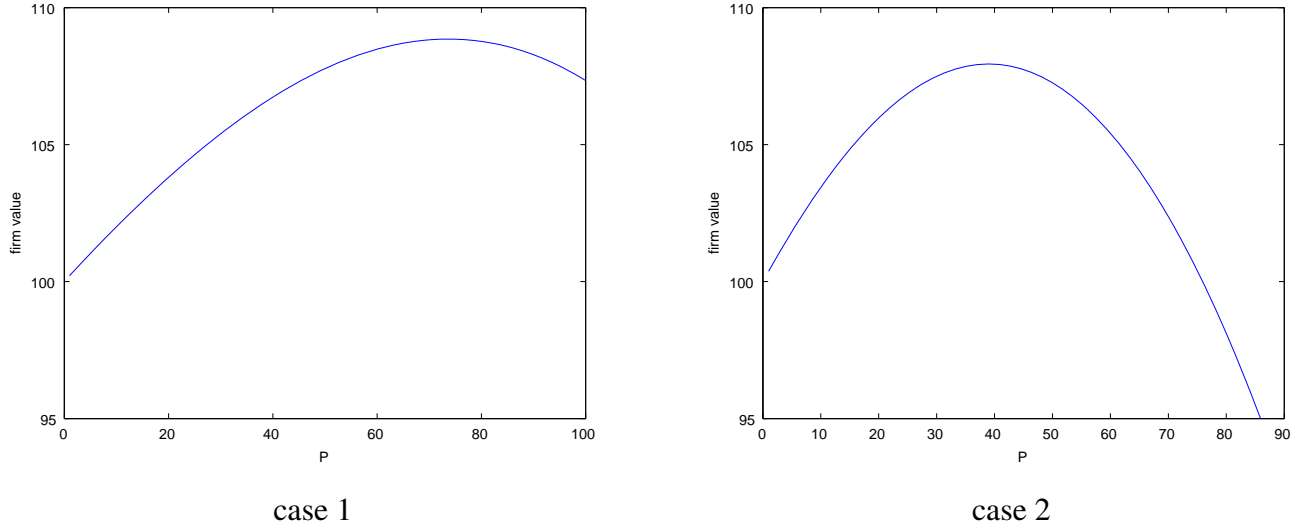


FIGURE 4. The firm value as a function of  $P$  for the two-stage problem.

Figure 2 shows the function  $K_1^{(r+m)}$  in (3.9) as a function of  $B$  when  $P = 50$ . As shown in the proof of Proposition 4.1, this is indeed monotonically increasing for both cases. The optimal bankruptcy level  $B^*$  can therefore be computed by the bisection method and we obtain  $B^* = 3.61$  and  $B^* = 3.64$  for cases 1 and 2, respectively. In case 2, we notice a non-smooth point at zero and this is caused because  $c = 0$  in the definition of  $f_2$  in (5.3). However, because  $B^*$  is chosen larger than zero, the tax rate becomes constant until bankruptcy. Furthermore, because  $B^* < b$ , the loss fraction  $\bar{\eta}$  at bankruptcy ends up being a constant. In case 1, on the other hand, the tax rate fluctuates over time and  $\bar{\eta}$  is also asset-value-dependent.

Using the optimal bankruptcy levels  $B^*$  computed above, we can compute the equity/debt/firm values. In Figure 3, we plot their values as a function of asset value  $V_0 = e^x \geq e^B$  for  $B = B^* - 0.2, B^* - 0.1, B^*, B^* + 0.1, B^* + 0.2$ . As shown in Lemma 3.4, we can confirm that, when  $B$  is taken lower than  $B^*$ , it violates the limited liability constraint (2.5). For  $B$  larger than  $B^*$ , the equity value is dominated by the value under  $B^*$ . We note that continuous fit at  $B$  always holds as in (3.13). Also, as we have discussed in Remark 3.5, we also observe that smooth fit holds at  $B^*$ .

For the optimal capital structure, we solve the *two-stage problem* as studied by [8, 20, 21] where the final goal is to choose  $P$  (with  $m$  fixed constant) such that the firm's value  $\mathcal{V}$  is maximized, namely, for fixed  $x$ ,

$$(5.5) \quad \max_P \mathcal{V}(x; B^*(P), P)$$

where we emphasize the dependency of  $\mathcal{V}$  and  $B^*$  on  $P$ . As discussed in [8], this is only a rough approximation of the optimal capital structure/default strategy and does not fully capture the conflict of interest between debt holders and equity holders. However, this is commonly used, and one major

advantage of using this formulation is that (5.5) typically admits a global optimum  $P^*$ . In particular, the concavity of  $\mathcal{V}(x; B^*(P), P)$  with respect to  $P$  has been analytically shown by [8] for the double exponential jump diffusion case in their setting. In our case with a generalized bankruptcy cost and tax rate functions, the proof of concavity is significantly difficult. However, from our numerical results given below, it seems to hold at least with our specifications (5.1)-(5.3); see Figure 4 below.

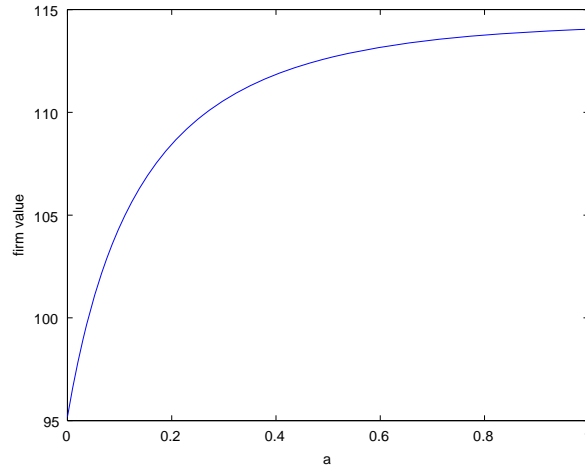
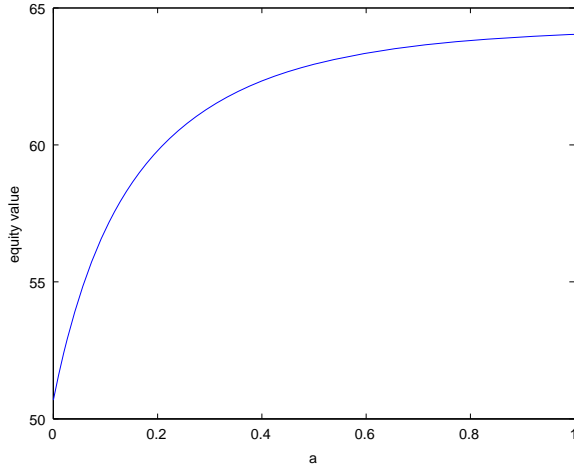
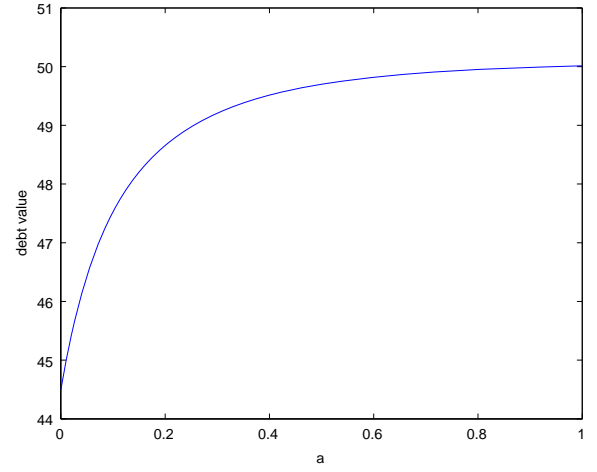
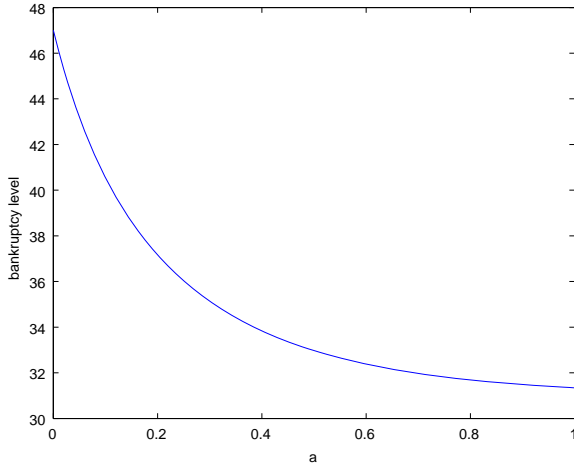
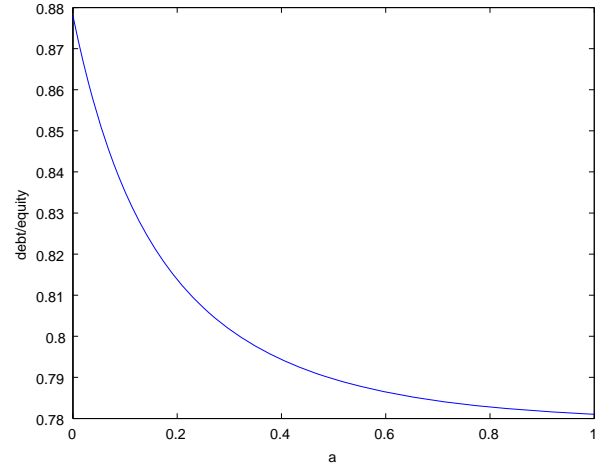
We set  $V_0 = 100$  (or  $x = \log(100)$ ) and obtain  $B^*$  for  $P$  running from 0 to 100. The corresponding firm value  $\mathcal{V}$  is computed for each  $P$  and  $B^* = B^*(P)$ , and is shown in Figure 4. As can be seen easily, these are indeed concave and hence the optimal face values of debt  $P^* = 73.7$  and  $P^* = 39$  are obtained for cases 1 and 2, respectively.

**5.1. Numerical results on the scale effects.** We are now ready to study the impacts of scale effects on the optimal bankruptcy levels, equity/debt/firm values as well as the optimal capital structure. We use the degree of bankruptcy cost concavity  $a$  and tax convexity  $c$  in (5.1)-(5.3) as proxies for scale effects and study how these values change with  $a$  and  $c$ . The parameters for (5.1)-(5.3) are the same as case 1 above, unless specified otherwise. We again set  $V_0 = 100$  (or  $x = \log(100)$ ).

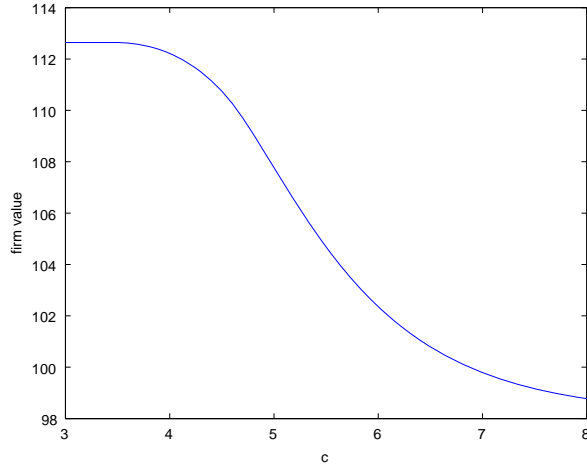
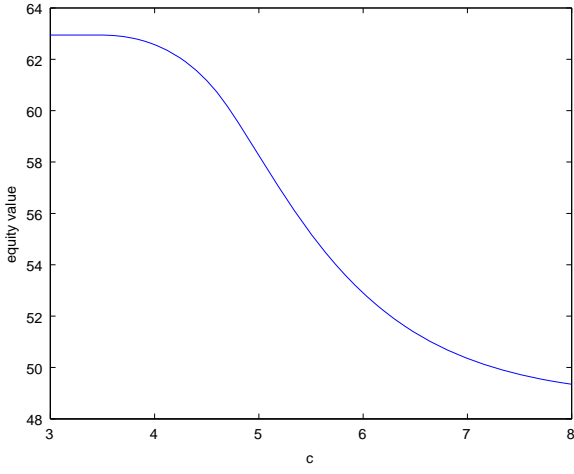
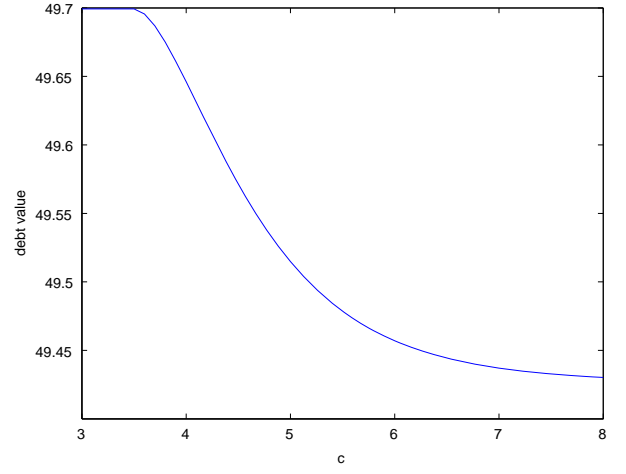
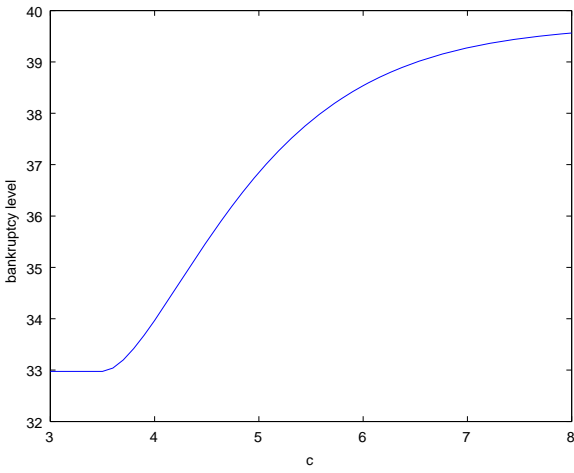
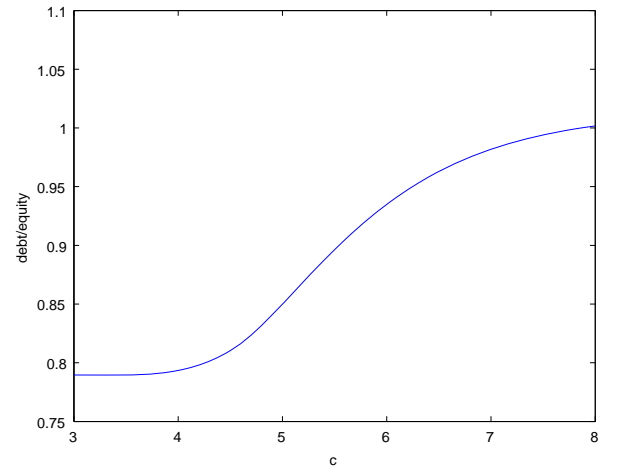
Firstly, we fix the face value  $P = 50$  and study the effects of bankruptcy cost concavity and tax convexity in Figures 5 and 6, respectively.

Figure 5 shows the impacts of the bankruptcy cost concavity. All the figures are with respect to  $a$  running from 0 to 1. Recalling that the bankruptcy cost is decreasing in  $a$ , most of the results are relatively straightforward. As can be observed in (i)-(iii), each of the firm/equity/debt values is increasing. From (iv), we see that the bankruptcy level  $B^*$  is decreasing. This is because, as the bankruptcy cost decreases (or  $a$  increases), the bankruptcy level can be lowered further without violating the limited liability constraint. Less obvious is the debt/equity ratio  $\mathcal{D}(x; B^*)/\mathcal{E}(x; B^*)$  as a function of  $a$  as shown in (v). While both the equity and debt values are increasing in  $a$ , we observe that the former increases faster than the latter. Notice that all the figures here exhibit not only monotonicity but also concavity/convexity.

Figure 6 shows the impacts of the tax convexity. All the figures are with respect to  $c$  running from 3 to 8. Because it reduces the tax benefits monotonically, the results here are essentially opposite of those in Figure 5. Each of the firm/equity/debt values is decreasing. The value of  $B^*$  is increasing because it must be raised so as not to violate the limited liability constraint. The debt/equity ratio is increasing; while the equity value is directly affected by the reduction of tax benefits (or the reduction in the asset value), the debt value is diminished only indirectly by the change of  $B^*$ . Contrary to Figure 5, these figures no longer exhibit concavity nor convexity. This is due to the non-concave/convex nature of the tax rate function (5.3), which converges to zero as  $c$  gets smaller and to the maximum tax benefit rate  $P\hat{\gamma}\hat{\rho}$  as it gets larger. This means that the marginal effect of changing the value of  $c$  vanishes as  $c$  gets sufficiently large or sufficiently small. In particular, for  $c$  smaller than the bankruptcy level  $B^*$ , because  $f_2(x)$  is constant uniformly on  $[c, \infty)$ , changing the value of  $c$  does not have any effect because  $f_2(X_t)$  remains constant at any time before bankruptcy. This results in a flat line on each figure for small  $c$ .

(i) firm value  $\mathcal{V}(x; B^*)$ (ii) equity value  $\mathcal{E}(x; B^*)$ (iii) debt value  $\mathcal{D}(x; B^*)$ (iv) bankruptcy level  $e^{B^*}$ (v) debt/equity ratio  $\mathcal{D}(x; B^*)/\mathcal{E}(x; B^*)$ FIGURE 5. Effects of bankruptcy cost concavity  $a$  for fixed face value  $P = 50$ .



(i) firm value  $\mathcal{V}(x; B^*)$ (ii) equity value  $\mathcal{E}(x; B^*)$ (iii) debt value  $\mathcal{D}(x; B^*)$ (iv) bankruptcy level  $e^{B^*}$ (v) debt/equity ratio  $\mathcal{D}(x; B^*)/\mathcal{E}(x; B^*)$ FIGURE 6. Effects of tax convexity  $c$  for fixed face value  $P = 50$ .

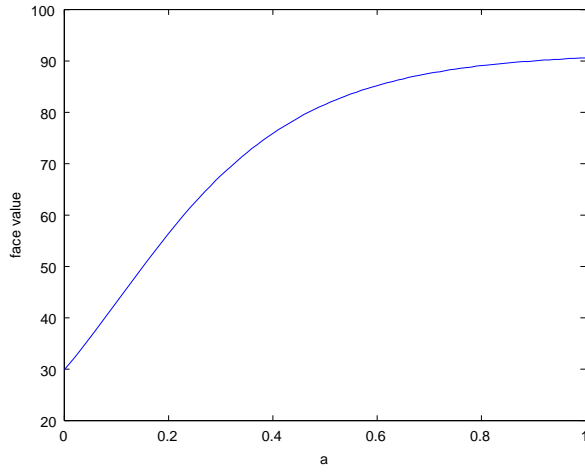
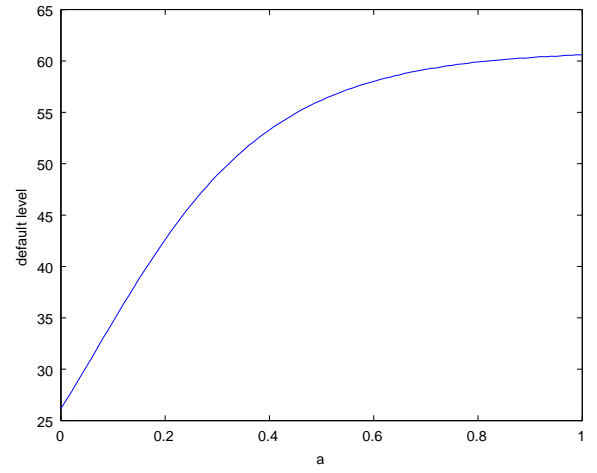
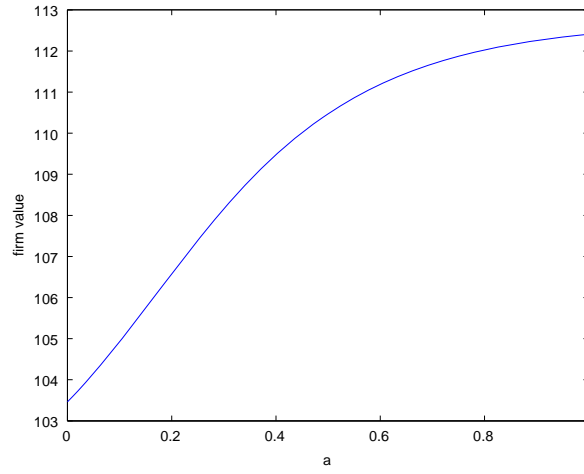
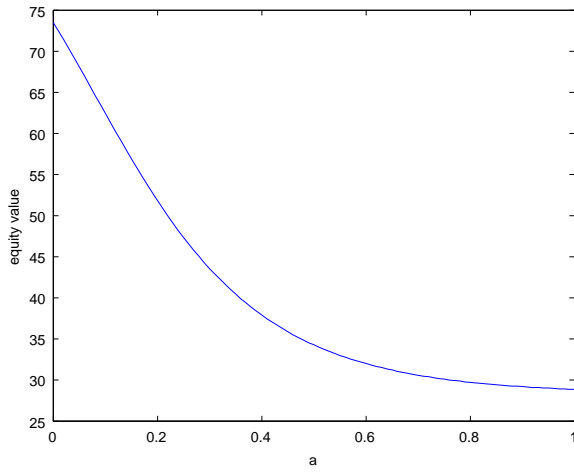
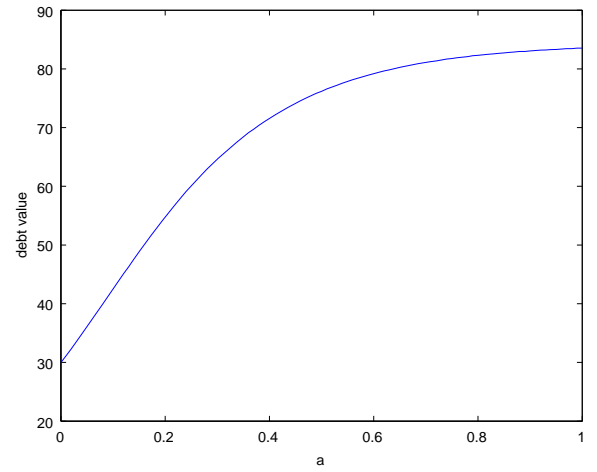
(i) optimal face value  $P^*(a)$ (ii) bankruptcy level  $e^{B^*}$  at  $(a, P^*(a))$ (iii) firm value at  $(a, P^*(a))$ (iv) equity value at  $(a, P^*(a))$ (v) debt value at  $(a, P^*(a))$ 

FIGURE 7. Effects of bankruptcy cost concavity in the two-stage problem.

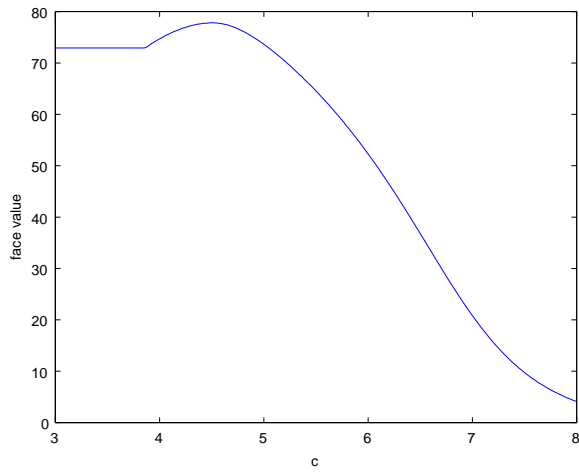
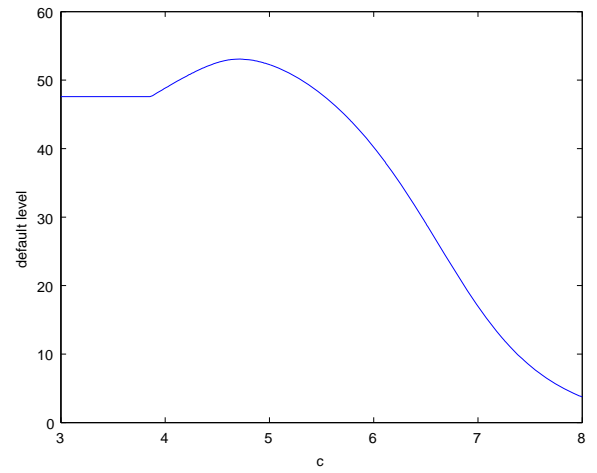
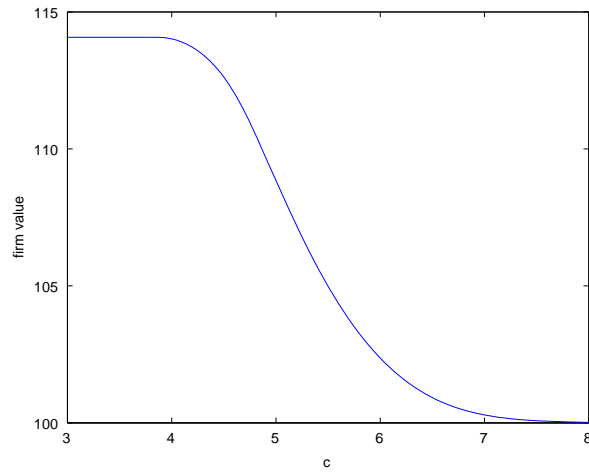
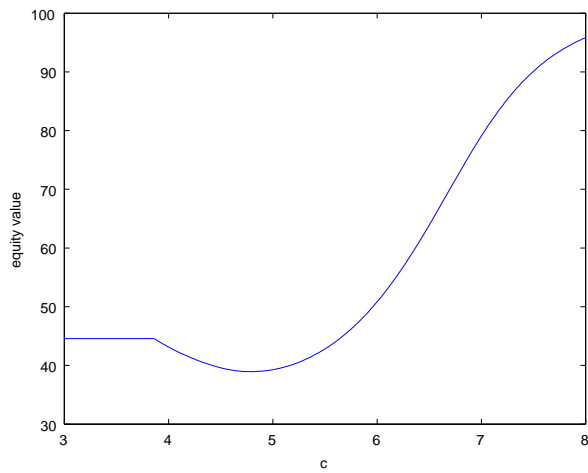
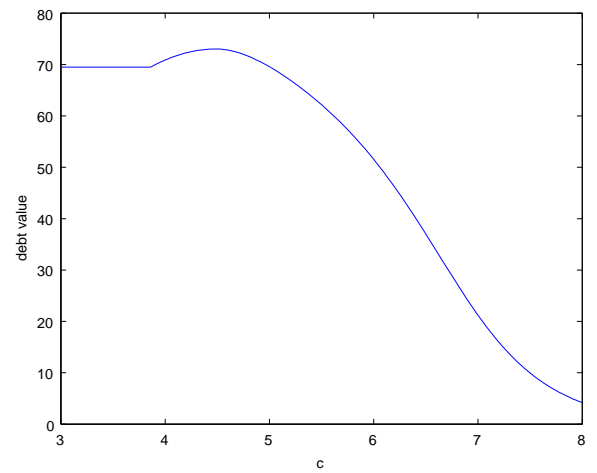
(i) optimal face value  $P^*(c)$ (ii) bankruptcy level  $e^{B^*}$  at  $(c, P^*(c))$ (iii) firm value at  $(c, P^*(c))$ (iv) equity value at  $(c, P^*(c))$ (v) debt value at  $(c, P^*(c))$ 

FIGURE 8. Effects of tax convexity in the two-stage problem.

We now move on to studying the impacts of the scale effects on the optimal capital structure. We consider the two-stage problem (5.5) and let the face value  $P$  be a control variable.

Figure 7 shows the impacts of the bankruptcy cost concavity on the optimal capital structure. For each value of  $a$  ranging from 0 to 1, we solve the two-stage problem (5.5) and compute (i) the optimal face value  $P^*(a)$  as well as (ii) bankruptcy level, and (iii)-(v) firm/equity/debt values corresponding to  $(a, P^*(a))$ . From (i), we see that  $P^*(a)$  is increasing. This is because, as the bankruptcy cost decreases, the firm tends to rely more on debt financing so as to enjoy more tax benefits. While the problem is complicated due to the (conflicting) inner optimization where the equity holder chooses the bankruptcy level to maximize the equity value, it turns out that the (firm) value of the two-stage problem is monotonically decreasing in  $a$  as seen in (iii). It can be confirmed that by choosing optimally the face value  $P^*$ , the firm value is necessarily no less than the unlevered asset value  $V_0 = 100$  (which is attained by setting  $P = 0$ ). Notice that it is not always true when the face value is fixed as in Figure 5-(i). The results in Figure 7 are different from those in Figure 5 because the face value also changes as  $a$  changes. Nonetheless, we still observe monotonicity and convexity/concavity in these figures.

Figure 8 shows the impacts of tax convexity. Similarly to Figure 7, for each value of  $c$  ranging from 3 to 8, we compute the optimal face value  $P^*(c)$  and the values corresponding to  $(c, P^*(c))$ . Interestingly, the optimal face value  $P^*(c)$  fails to be monotone and, for this reason, the bankruptcy level and equity/debt values also fail to be monotone. Nevertheless, the firm value, or the value of the two-stage problem, is still monotonically decreasing. The non-monotonicity of  $P^*(c)$  is caused due to the non-concave/convex nature of the tax benefits. As in the case of Figure 6, the marginal effect of changing  $c$  is negligible when  $c$  is sufficiently large or sufficiently small. For a medium value of  $c$ , we observe the monotonicity, which is consistent with the intuition that the tax convexity decreases the tax benefits and hence also the leverage. However, for a neighborhood of  $c$  where the marginal effect is very small but not exactly zero, non-monotonicity of the optimal face value can happen due to the two conflicting maximizations of the equity value and the firm value. Also recall that the tax benefits vanish as  $c$  increases. We can confirm that, as  $c$  increases, the optimal face value  $P^*(c)$  converges to zero, thereby making the debt worthless, and the firm value converges to the unlevered asset value  $V_0 = 100$ .

In summary, the scale effects have significant impacts on the capital structure. Depending on the choice of bankruptcy cost and tax rate functions, the results can also become very complicated. The computation we conducted here is efficient and does not rely on heavy algorithms. This is partly due to the form of its scale function (see Appendix A) and also to the choice of  $\eta$ ,  $f_1$  and  $f_2$  given above. However, this can be extended very easily to the hyperexponential case; see [9]. For other spectrally negative Lévy processes with explicit forms of scale functions, see [14, 16, 17]. We also remark that the solutions can in principle be computed numerically for any choice of spectrally negative Lévy process by using the approximation algorithms of the scale function such as [9, 23].

## 6. CONCLUDING REMARKS

We have studied a generalization of the Leland-Toft optimal capital structure model where the values of bankruptcy costs and tax benefits are dependent on the asset value. Focusing on the spectrally negative Lévy model, we obtained a sufficient condition for optimality, which holds under some assumptions that have been justified empirically. The solutions admit semi-explicit forms in terms of the scale function and allow for instant computation of the optimal capital structure. The generalization we achieved in this paper can be applied to realize more flexible models to derive optimal capital structures.

For future research, it is most natural to consider its extension to a general Lévy case with two-sided jumps. It is expected that the same generalization can be achieved at least for the Lévy processes admitting rational forms of Wiener-Hopf factorization such as double exponential jump diffusion [8] and, more generally, the phase-type Lévy process [2]. It is also beneficial to obtain other sufficient conditions for optimality that we have not covered in Section 4. Finally, a model has to adapt to a changing economic environment and financial restrictions. An interesting but nonetheless challenging extension would be to change the limited liability constraint so that the equity value must be bounded from below by some positive constant as opposed to be kept simply above zero as in the current model.

## APPENDIX A. THE EQUITY/DEBT/FIRM VALUES FOR SECTION 5

In this appendix, we obtain explicit expressions of the function  $K_1^{(r,m)}$  as well as the equity/debt/firm values that are used in Section 5.

First, the scale function for the process with  $\sigma > 0$  and Lévy measure (5.4) admits an explicit form. For every  $q > 0$ , there are two negative real roots  $-\xi_{1,q}$  and  $-\xi_{2,q}$  to the equation  $\kappa(s) = q$  and, as is discussed in [9], its scale function is given by

$$(A.1) \quad \begin{aligned} W^{(q)}(x) &= \sum_{i=1,2} C_{i,q} [e^{\Phi(q)x} - e^{-\xi_{i,q}x}], \\ Z^{(r)}(x) &= 1 + r \sum_{i=1,2} C_{i,q} \left[ \frac{1}{\Phi(r)} (e^{\Phi(r)x} - 1) + \frac{1}{\xi_{i,r}} (e^{-\xi_{i,r}x} - 1) \right], \end{aligned}$$

for some positive constants  $C_{1,q}$  and  $C_{2,q}$ ; see [9] for their explicit expressions.

For the computation of  $K_1^{(r,m)}$  in (3.9), straight algebra obtains, for every  $B \in \mathbb{R}$  and  $q > 0$ ,

$$(A.2) \quad G_1^{(q)}(B) = \frac{P(\hat{\rho} + m)}{\Phi(q)} \quad \text{and} \quad G_2^{(q)}(B) = P\hat{\gamma}\hat{\rho} \left[ e^{B-c} \frac{1 - e^{-(\Phi(q)-1)[(c-B) \vee 0]}}{\Phi(q) - 1} + \frac{1}{\Phi(q)} e^{-\Phi(q)[(c-B) \vee 0]} \right],$$

and

$$(A.3) \quad H^{(q)}(B) = \lambda\eta(B) \left( \frac{1}{\Phi(q) + \beta} \right) - e^B Q(B; \Phi(q), \infty),$$

where

$$Q(B; \zeta, l) := \int_0^\infty \Pi(du) \int_0^{u \wedge l} e^{-(\zeta-1)z-u} \bar{\eta}(B-u+z) dz, \quad \zeta, l > 0.$$

Here, it can be shown, for any  $\zeta \in \mathbb{R}$  and  $l > 0$  with  $\tilde{b} = (B-b) \vee 0$ ,

$$\begin{aligned} Q(B; \zeta, l)/\eta_0 &= \frac{\lambda\beta}{\zeta-1} \left[ \left( \frac{e^{-\tilde{b}(1+\beta)}}{1+\beta} (1 - e^{-l(1+\beta)}) - \frac{e^{-\tilde{b}(\zeta+\beta)+(\zeta-1)\tilde{b}}}{\zeta+\beta} (1 - e^{-l(\zeta+\beta)}) \right) \right. \\ &\quad \left. + \frac{1}{1+\beta} e^{-(l+\tilde{b})(1+\beta)} (1 - e^{-(\zeta-1)l}) \right] \\ &\quad + \frac{\lambda\beta}{\zeta-1+a} \left[ \frac{1}{\beta+1-a} (e^{-a\tilde{b}} - e^{-\tilde{b}(\beta+1)}) + \frac{1}{\zeta+\beta} (e^{-\tilde{b}(\beta+1)} - e^{-(\tilde{b}+l)(\zeta+\beta)+\tilde{b}(\zeta-1)}) \right. \\ &\quad \left. + \frac{1}{\beta+1-a} e^{-l(\beta+\zeta)-\tilde{b}(\beta+1-a)-a\tilde{b}} \right] \\ &\quad - \frac{\lambda\beta e^{-a\tilde{b}}}{\zeta-1+a} \left[ \frac{1}{\zeta+\beta} (1 - e^{-l(\zeta+\beta)}) + \frac{1}{1-a+\beta} e^{-l(\zeta+\beta)} \right]. \end{aligned}$$

For the equity/debt/firm values, we need  $\mathcal{M}_i^{(q)}(x; B)$  and  $\Lambda^{(q)}(x; B)$  in (3.4). For the former, for every  $q > 0$  and  $x > B$ ,

$$\int_B^x W^{(q)}(x-y) f_1(y) dy = (P\hat{\rho} + p) \sum_{i=1,2} C_{i,q} \left[ \frac{1}{\Phi(q)} (e^{\Phi(q)(x-B)} - 1) - \frac{1}{\xi_{i,q}} (1 - e^{-\xi_{i,q}(x-B)}) \right]$$

and

$$\begin{aligned} \int_B^x W^{(q)}(x-y) f_2(y) dy &= P\hat{\rho}\gamma \sum_{i \in 1,2} C_{i,q} \\ &\times \left[ e^{-c} \left[ \frac{e^{\Phi(q)x}}{\Phi(q)-1} (e^{-(\Phi(q)-1)B} - e^{-(\Phi(q)-1)(x \wedge c \vee B)}) - \frac{e^{-\xi_{i,q}x}}{\xi_{i,q}+1} (e^{(\xi_{i,q}+1)(x \wedge c \vee B)} - e^{(\xi_{i,q}+1)B}) \right] \right. \\ &\quad \left. + \left[ \frac{1}{\Phi(q)} (e^{\Phi(q)(x-x \wedge c \vee B)} - 1) - \frac{1}{\xi_{i,q}} (1 - e^{-\xi_{i,q}(x-x \wedge c \vee B)}) \right] \right]. \end{aligned}$$

These together with (A.1)-(A.2) obtain  $\mathcal{M}_i^{(q)}(x; B)$  for any  $i = 1, 2$  and  $x > B$ .

For  $\Lambda^{(q)}(x; B)$ , some algebra shows

$$\begin{aligned} \int_0^\infty \Pi(du) \int_0^{u \wedge (x-B)} W^{(q)}(x-z-B) dz \\ = \lambda \sum_{i=1,2} C_{i,q} \left[ \frac{1}{\Phi(q)+\beta} (e^{\Phi(q)(x-B)} - e^{-\beta(x-B)}) + \frac{1}{\xi_{i,q}-\beta} (e^{-\xi_{i,q}(x-B)} - e^{-\beta(x-B)}) \right] \end{aligned}$$

and

$$\begin{aligned} & \int_0^\infty \Pi(du) \left[ \int_0^{u \wedge (x-B)} W^{(q)}(x-z-B) \eta(z+B-u) dz \right] \\ &= \sum_{i=1,2} C_{i,q} e^{\Phi(q)(x-B)+B} Q(B; \Phi(q), x-B) - \sum_{i=1,2} C_{i,q} e^{-\xi_{i,q}(x-B)+B} Q(B; -\xi_{i,q}, x-B). \end{aligned}$$

Now  $\Lambda^{(q)}(x; B)$  is immediately obtained by (A.1) and (A.3).

## APPENDIX B. PROOFS

*Proof of Lemma 3.2.* By (3.5) and (3.7), the left-hand side equals

$$\begin{aligned} e^B (\Gamma^{(r+m)}(x-B) - \Gamma^{(r+m)'}(x-B)) &= -e^B \left[ \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} W^{(r+m)'}(x-B) \right. \\ &\quad \left. + (\kappa(1) - (r+m)) W^{(r+m)}(x-B) - \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} W^{(r+m)}(x-B) \right], \end{aligned}$$

which equals the right-hand side.  $\square$

*Proof of Proposition 3.1.* Applying Lemmas 3.1-3.2 in (3.6),

$$\begin{aligned} \frac{\partial}{\partial B} \mathcal{E}(x; B) &= -\Theta^{(r+m)}(x-B) \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} e^B \\ &\quad + \Theta^{(r+m)}(x-B) \left[ G_1^{(r+m)}(B) + \frac{r+m}{\Phi(r+m)} \eta(B) + H^{(r+m)}(B) + \frac{\sigma^2}{2} \eta'(B) \right] \\ &\quad - \Theta^{(r)}(x-B) \left[ G_2^{(r)}(B) + \frac{r}{\Phi(r)} \eta(B) + H^{(r)}(B) + \frac{\sigma^2}{2} \eta'(B) \right] \\ &= -\Theta^{(r+m)}(x-B) \frac{\kappa(1) - (r+m)}{1 - \Phi(r+m)} e^B \\ &\quad + \Theta^{(r+m)}(x-B) \left[ G_1^{(r+m)}(B) + \frac{r+m}{\Phi(r+m)} \eta(B) + H^{(r+m)}(B) + \frac{\sigma^2}{2} \eta'(B) \right] \\ &\quad - \Theta^{(r+m)}(x-B) \left[ G_2^{(r)}(B) + \frac{r}{\Phi(r)} \eta(B) + H^{(r)}(B) + \frac{\sigma^2}{2} \eta'(B) \right] \\ &\quad - (\Theta^{(r)}(x-B) - \Theta^{(r+m)}(x-B)) \left[ G_2^{(r)}(B) + \frac{r}{\Phi(r)} \eta(B) + H^{(r)}(B) + \frac{\sigma^2}{2} \eta'(B) \right], \end{aligned}$$

which matches (3.10).  $\square$

*Proof of Remark 3.4.* Because for any  $q > 0$

$$\frac{q}{\Phi(q)} = c + \frac{1}{2} \sigma^2 \Phi(q) + \int_0^\infty \Pi(du) \left( \frac{e^{-\Phi(q)u} - 1}{\Phi(q)} + u 1_{\{u \in (0,1)\}} \right),$$



we obtain

$$\begin{aligned} \frac{r+m}{\Phi(r+m)} - \frac{r}{\Phi(r)} &= \frac{1}{2}\sigma^2 (\Phi(r+m) - \Phi(r)) - \int_0^\infty \Pi(du) \left( \frac{e^{-\Phi(r)u} - 1}{\Phi(r)} + u1_{\{u \in (0,1)\}} \right) \\ &\quad + \int_0^\infty \Pi(du) \left( \frac{e^{-\Phi(r+m)u} - 1}{\Phi(r+m)} + u1_{\{u \in (0,1)\}} \right) \\ &= \frac{1}{2}\sigma^2 (\Phi(r+m) - \Phi(r)) + \int_0^\infty \Pi(du) \left( \frac{1 - e^{-\Phi(r)u}}{\Phi(r)} - \frac{1 - e^{-\Phi(r+m)u}}{\Phi(r+m)} \right). \end{aligned}$$

Substituting this in (3.8), we obtain the result.  $\square$

*Proof of Lemma 3.3.* The result for the case  $\sigma > 0$  is clear because, by (3.3),  $W^{(q)}(0) = 0$  and  $W^{(q)'}(0+) = 2/\sigma^2$  for any  $q > 0$ . Suppose  $\sigma = 0$ . Then as in the proof of Lemma 4.4 of [17], we have

$$\begin{aligned} \lim_{x \downarrow 0} [W^{(r)'}(x) - W^{(r+m)'}(x)] &= \lim_{\lambda \uparrow \infty} \int_0^\infty \lambda e^{-\lambda x} [W^{(r)'}(x) - W^{(r+m)'}(x)] dx \\ &= \lim_{\lambda \uparrow \infty} \left[ \frac{\lambda^2}{\kappa(\lambda) - r} - \frac{\lambda^2}{\kappa(\lambda) - (r+m)} \right] = -m \lim_{\lambda \uparrow \infty} \left[ \frac{\lambda}{\kappa(\lambda) - r} \frac{\lambda}{\kappa(\lambda) - (r+m)} \right] = 0. \end{aligned}$$

Here the last equality holds because for any  $q > 0$

$$\frac{\kappa(\lambda) - q}{\lambda} = c + \int_{(0,\infty)} \left( \frac{e^{-\lambda x} - 1}{\lambda} + x1_{\{0 < x < 1\}} \right) \Pi(dx) - \frac{q}{\lambda} \xrightarrow{\lambda \uparrow \infty} \infty,$$

due to Fatou's lemma and  $\int_{(0,1)} x \Pi(dx) = \infty$ . This together with  $W^{(r)}(0) = W^{(r+m)}(0) = 0$  shows the result.  $\square$

*Proof of Lemma 3.4.* (i) Suppose  $X$  is of bounded variation and fix  $\widehat{B} < B^*$ . By (3.14), we have  $K_1^{(r,m)}(\widehat{B}) < 0$ . But by (3.13), this implies  $\mathcal{E}(\widehat{B}+; \widehat{B}) < 0$ , violating (2.5). Therefore, those  $B$  satisfying (2.5) must lie on  $[B^*, \infty)$ .

(ii) Suppose  $X$  is of unbounded variation and again fix  $\widehat{B} < B^*$ . For any sufficiently small  $\delta > 0$  such that  $K_1^{(r,m)}(x) < 0$  for every  $\widehat{B} < x < \widehat{B} + \delta$ , we have by Proposition 3.1

$$\begin{aligned} \inf_{\widehat{B} \leq x \leq \widehat{B} + \delta, \widehat{B} \leq y \leq x} \left. \frac{\partial}{\partial B} \mathcal{E}(x; B) \right|_{B=y} &\geq \inf_{0 \leq y \leq \delta} \Theta^{(r+m)}(y) \inf_{\widehat{B} \leq B \leq \widehat{B} + \delta} |K_1^{(r,m)}(B)| \\ &\quad - \sup_{0 \leq y \leq \delta} \{\Theta^{(r)}(y) - \Theta^{(r+m)}(y)\} \sup_{\widehat{B} \leq B \leq \widehat{B} + \delta} |K_2^{(r)}(B)|. \end{aligned}$$

This converges to some strictly positive value as  $\delta \downarrow 0$  by Lemma 3.3, (3.3) and (3.7); namely there exists  $\delta_0 > 0$  such that

$$\inf_{\widehat{B} \leq x \leq \widehat{B} + \delta_0, \widehat{B} \leq y \leq x} \left. \frac{\partial}{\partial B} \mathcal{E}(x; B) \right|_{B=y} > 0.$$

But by (3.13),  $\mathcal{E}(\widehat{B} + \delta_0; \widehat{B} + \delta_0) = 0$ , implying  $\mathcal{E}(\widehat{B} + \delta_0; \widehat{B}) < 0$ , violating (2.5). Therefore the proof is complete by contradiction.  $\square$

*Proof of Lemma 4.1.* Define, as the Laplace exponent of  $X$  under  $\mathbb{P}_1$  with the change of measure  $\frac{d\mathbb{P}_1}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = e^{X_t - \kappa(1)t}$ ,

$$\kappa_1(\beta) := \left( \sigma^2 + c - \int_{(0,1)} u(e^{-u} - 1) \Pi(du) \right) \beta + \frac{1}{2} \sigma^2 \beta^2 + \int_{(0,\infty)} (e^{-\beta u} - 1 + \beta u 1_{\{u \in (0,1)\}}) e^{-u} \Pi(du).$$

Then,  $\kappa_1(\Phi(q) - 1) = \kappa(\Phi(q)) - \kappa(1) = q - \kappa(1)$  for any  $q > 0$ ; see page 215 of [15]. This shows

$$\begin{aligned} \frac{\kappa(1) - r}{1 - \Phi(r)} &= \frac{\kappa_1(\Phi(r) - 1)}{\Phi(r) - 1} = \sigma^2 + c - \int_{(0,1)} u(e^{-u} - 1) \Pi(du) \\ &\quad + \frac{1}{2} \sigma^2 (\Phi(r) - 1) + \int_{(0,\infty)} \left( \frac{e^{-(\Phi(r)-1)u} - 1}{\Phi(r) - 1} + u 1_{\{u \in (0,1)\}} \right) e^{-u} \Pi(du) \end{aligned}$$

and

$$\begin{aligned} \frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} &= \frac{\kappa_1(\Phi(r + m) - 1)}{\Phi(r + m) - 1} = \sigma^2 + c - \int_{(0,1)} u(e^{-u} - 1) \Pi(du) \\ &\quad + \frac{1}{2} \sigma^2 (\Phi(r + m) - 1) + \int_{(0,\infty)} \left( \frac{e^{-(\Phi(r+m)-1)u} - 1}{\Phi(r + m) - 1} + u 1_{\{u \in (0,1)\}} \right) e^{-u} \Pi(du). \end{aligned}$$

Subtracting the former from the latter, we obtain the result.  $\square$

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